# XIX. *A Tenth Memoir on Quantics.*

*By* A. Cayley, *F.R.S., Sadlerian Professor of Pure Mathematics in the University of Cambridge.*

Received June 12,—Read June 20, 1878.

The present Memoir, which relates to the binary quintic $( * \between x, y)^5$, has been in hand for a considerable time: the chief subject-matter was intended to be the theory of a canonical form discovered by myself and which is briefly noticed in Salmon's 'Higher Algebra,' 3rd Ed. (1876), pp. 217, 218; writing $a, b, c, d, e, f, g \ldots u, v, w$ to denote the 23 *covariants* of the quintic, then $a, b, c, d, f$ are connected by the relation $f^2 = -a^3d + a^2bc - 4c^3$; and the form contains these covariants thus connected together, and also $e$; it, in fact, is $(1, 0, c, f, a^2b - 3c^2, a^2e - 2cf \between x, y)^5$.

But the whole plan of the Memoir was changed by Sylvester's discovery of what I term the Numerical Generating Function (N.G.F.) of the covariants of the quintic, and my own subsequent establishment of the Real Generating Function (R.G.F.) of the same covariants. The effect of this was to enable me to establish for any given degree in the coefficients and order in the variables, or as it is convenient to express it, for any given deg-order whatever, a selected system of powers and products of the covariants, say a system of "segregates:" these are asyzygetic, that is, not connected together by any linear equation with numerical coefficients; and they are also such that every other combination of covariants of the same deg-order, say every "congregate" of the same deg-order, can be expressed (and that, obviously, in one way only) as a linear function, with numerical coefficients, of the segregates of that deg-order. The number of congregates of a given deg-order is precisely equal to the number of the independent syzygies of the same deg-order, so that these syzygies give in effect the congregates in terms of the segregates: and the proper form in which to exhibit the syzygies is thus to make each of them give a single congregate in terms of the segregates: viz., the left hand side can always be taken to be a monomial congregate $a^\alpha b^\beta \ldots$ or, to avoid fractions, a numerical multiple of such form; and the right hand side will then be a linear function, with numerical coefficients, of the segregates of the same deg-order. Supposing such a system of syzygies obtained for a given deg-order, any covariant function (rational and integral function of covariants) is at once expressible as a linear function of the segregates of that deg-order: it is in fact only necessary to substitute therein for every monomial congregate its value as a linear function of the segregates. Using the word covariant in its most general sense, the conclusion thus is that every covariant can be expressed, and that in one way only, as a linear function of segregates, or say in the segregate form.

Reverting to the theory of the canonical form, and attending to the relation $f^2=-a^3d+a^2bc-4c^3$, it thereby appears that every covariant multiplied by a power of the quintic itself $a$, can be expressed, and that in one way only, as a rational and integral function of the covariants $a$, $b$, $c$, $d$, $e$, $f$, linear as regards $f$: say every covariant multiplied by a power of $a$ can be expressed, and that in one way only, in the "standard" form: as an illustration take $a^2h=6acd+4bc^2+ef$. Conversely an expression of the standard form, that is, a rational and integral function of $a$, $b$, $c$, $d$, $e$, $f$, linear as regards $f$, not explicitly divisible by $a$, may very well be really divisible by a power of $a$ (the expression of the quotient of course containing one or more of the higher covariants $g$, $h$, &c.), and we say that in this case the expression is divisible, and has for its divided form the quotient expressed as a rational and integral function of covariants. Observe that in general the divided form is not perfectly definite, only becoming so when expressed in the before-mentioned segregate form, and that this further reduction ought to be made: there is occasion, however, to consider these divided forms, whether or not thus further reduced, and moreover it sometimes happens that the non-segregate form presents itself, or can be expressed, with integer numerical coefficients, while the coefficients of the corresponding segregate form are fractional.

The canonical form is peculiarly convenient for obtaining the expressions of the several derivatives (GORDAN'S *Uebereinanderschiebungen*) $(a, b)^1$, $(a, b)^2$, &c. (or as I propose to write them $ab1$, $ab2$, &c.), which can be formed with two covariants, the same or different, as rational and integral functions of the several covariants. It will be recollected that in GORDAN'S theory these derivatives are used in order to establish the system of the 23 covariants: but it seems preferable to have the system of covariants, and by means of them to obtain the theory of the derivatives.

I mention at the end of the Memoir two expressions (one or both of them due to SYLVESTER) for the N.G.F. of a binary sextic.

The several points above adverted to are considered in the Memoir; the paragraphs are numbered consecutively with those of the former Memoirs upon Quantics.

## *The Numerical and Real Generating Functions.*—Article Nos. 366 to 374, and Table No. 96.

366. I have in my Ninth Memoir (1871) given what may be called the Numerical Generating Function (N.G.F.) of the covariants of a quartic; this was

$$\mathrm{A}(x)=\frac{1-a^6x^{12}}{1-ax^4.\ 1-a^2x^4.\ 1-a^2.\ 1-a^3.\ 1-a^3x^6},$$

the meaning being that the number of asyzygetic covariants $a^\theta x^\mu$, of the degree $\theta$ in the coefficients and order $\mu$ in the variables, or say of the deg-order $\theta.\mu$ is equal to the coefficient of $a^\theta x^\mu$ in the development of this function. And I remarked that the formula indicated that the covariants were made up of $(ax^4, a^2x^4, a^2, a^3, a^3x^6)$ the quartic itself, the Hessian, the quadrinvariant, the cubinvariant, and the cubicovariant, these being connected by a syzygy $a^6x^{12}$ of the degree 6 and order 12. Calling these covariants $a, b, c, d, e$, so that these italic small letters stand for covariants,

| Deg-order. | |
|---|---|
| 1.4 | $a$ |
| 2.0 | $b$ |
| 2.4 | $c$ |
| 3.0 | $d$ |
| 3.6 | $e$ |

then it is natural to consider what may be called the Real Generating Function (R.G.F.): this is

$$\frac{1-e^2}{1-a\ .\ 1-b\ .\ 1-c\ .\ 1-d\ .\ 1-e};$$

the development of this contains, as it is easy to see, only terms of the form $a^\alpha b^\beta c^\gamma d^\delta$ and $a^\alpha b^\beta c^\gamma d^\delta e$, each with the coefficient $+1$, so that the number of terms of a given deg-order $\theta.\mu$ is equal to the coefficient of $a^\theta x^\mu$ in the first-mentioned function: and these terms of a given deg-order represent the asyzygetic covariants of that deg-order: any other covariant of the same deg-order is expressible as a linear function of them. For instance, deg-order 6.12, the terms of the R.G.F. are $a^3d$, $a^2bc$, $c^3$: there is one more term $e^2$ of the same deg-order; hence $e^2$ must be a linear function of these: and in fact $e^2=-a^3d+a^2bc-4c^3$, viz., this is the equation $\Phi^2=-\mathrm{U}^3\mathrm{J}+\mathrm{U}^2\mathrm{IH}-4\mathrm{H}^3$.

367. Sylvester obtained an expression for the N.G.F. of the quintic : this is

$$\frac{\begin{array}{rl}
a^0\,. & 1 \\
+a^3\,. & x^3+x^5+x^9 \\
+a^4\,. & x^4+x^6 \\
+a^5\,. & x\ +x^3+x^7-x^{11} \\
+a^6\,. & x^2+x^4 \\
+a^7\,. & x\ +x^5-x^9 \\
+a^8\,. & x^2+x^4 \\
+a^9\,. & x^3+x^5-x^7 \\
+a^{10}. & x^2+x^4-x^{10} \\
+a^{11}. & x\ +x^3-x^9 \\
+a^{12}. & x^2-x^8-x^{10} \\
+a^{13}. & x\ -x^7-x^9 \\
+a^{14}. & x^4-x^6-x^8 \\
+a^{15}. & -x^7-x^9 \\
+a^{16}. & x^2-x^6-x^{10} \\
+a^{17}. & -x^7-x^9 \\
+a^{18}. & 1\ -x^4-x^8-x^{10} \\
+a^{19}. & -x^5-x^7 \\
+a^{20}. & -x^2-x^6-x^8 \\
+a^{23}. & -x^{11}
\end{array}}{1-ax^5.\ 1-a^2x^2.\ 1-a^2x^6.\ 1-a^4.\ 1-a^8.\ 1-a^{12}}$$

viz., expanding this function in ascending powers of $a$, $x$, then if a term is $\mathrm{N}a^\theta x^\mu$, this means that there are precisely N asyzygetic covariants of the deg-order $\theta.\mu$.

368. It is known that the number of the irreducible covariants of the binary quintic is $=23$ ; representing these by the letters $a$, $b$, $c$, $d$, $e$, $f$, $g$, $h$, $i$, $j$, $k$, $l$, $m$, $n$, $o$, $p$, $q$, $r$, $s$, $t$, $u$, $v$, $w$ ($a$ the quintic itself), the deg-orders of these, and the references to the tables which give them are

| Deg-order. | | Tab. | Mem. |
|---|---|---|---|
| 1.5 | $a$ | | |
| 2.2 | $b$ | 13 | 2 |
| ,,.6 | $c$ | 14 | ,, |
| 3.3 | $d$ | 15 | ,, |
| ,,.5 | $e$ | 16 | ,, |
| ,,.9 | $f$ | 17 | ,, |
| 4.0 | $g$ | 18 | ,, |
| ,,.4 | $h$ | 19 | ,, |
| ,,.6 | $i$ | 20 | ,, |
| 5.1 | $j$ | 21 | ,, |
| ,,.3 | $k$ | 22 | ,, |
| ,,.7 | $l$ | 23 | ,, |
| 6.2 | $m$ | 83 | 8 |
| ,,.4 | $n$ | 84 | ,, |
| 7.1 | $o$ | 90 | 9 |
| ,,.5 | $p$ | 91 | ,, |
| 8.0 | $q$ | 25 | ,, |
| ,,.2 | $r$ | 92 | ,, |
| 9.3 | $s$ | * | |
| 11.1 | $t$ | 94 | 9 |
| 12.0 | $u$ | 29 | 3 |
| 13.1 | $v$ | 95 | 9 |
| 18.0 | $w$ | 29A | 5 |

Starting from the foregoing expression of the N.G.F. of the quintic, we can, instead of each term $a^{\theta}x^{\mu}$, introduce a covariant or product of covariants of the proper deg-order $\theta.\mu$: the mode of doing this depends of course on the different admissible partitions of $\theta$, $\mu$, and it is for some of the terms very indeterminate: for instance, $a^5x^{11}$ is $ai$, $bf$, or $ce$. *I found it possible to perform the whole process so as to satisfy a condition which will be presently referred to; and I found

---

* See end of Memoir. The S of Table 93 has the value $-96(\mathrm{D}, \mathrm{M})+16\,\mathrm{BO}-7\mathrm{GK}$, but it is better to use the simple value $-(\mathrm{D}, \mathrm{M})$; and the S of the present Memoir has this value, say $\mathrm{S}=-(d, m)$.

| R.G.F. of quintic= | | Deg-orders. |
|---|---|---|
| $1$ . | $1-b^5$ | 0.0—10.10 |
| $+d$ . | $1-ag^2$ | 3.3—12. 8 |
| $+e$ . | $1-b^3$ | 3.5— 7. 9 |
| $+f$ . | $1-b$ | 3.9— 5.11 |
| $+h$ . | $1-ag^2$ | 4.4—13. 9 |
| $+i$ . | $1-b^2g$ | 4.6—12.10 |
| $+j$ . | $1-ag^2$ | 5.1—14. 6 |
| $+k$ . | $1-b^2$ | 5.3— 9. 7 |
| $+l$ . | $1-bg$ | 5.7—11. 9 |
| $+m$ . | $1-ag^2$ | 6.2—15. 7 |
| $+n$ . | $1-b^2g$ | 6.4—14. 8 |
| $+o$ . | $1-b^3$ | 7.1—13. 7 |
| $+p$ . | $1-b^2g$ | 7.5—15. 9 |
| $+r$ . | $1-b^2g$ | 8.2—16. 6 |
| $+dj$ . | $1-ag^2$ | 8.4—17. 9 |
| $+s$ . | $1-abg$ | 9.3—16.10 |
| $+hj$ . | $1-ag^2$ | 9.5—18.10 |
| $+j^2$ . | $1-ag^2$ | 10.2—19. 7 |
| $+jk$ . | $1-b^2g$ | 10.4—18. 8 |
| $+t$ . | $1-b^3$ | 11.1—17. 7 |
| $+jm$ . | $1-ag^2$ | 11.3—20. 8 |
| $+jo$ . | $1-bg$ | 12.2—18. 4 |
| $+v$ . | $1-b^5$ | 13.1—23.11 |
| $+js$ . | $1-bg$ | 14.4—20. 6 |
| $+jt$ . | $1-g$ | 16.2—20. 2 |
| $+w$ . | $1-a$ | 18.0—19. 5 |

---

$$1-a\,.\,1-b\,.\,1-c\,.\,1-g\,.\,1-q\,.\,1-u$$

where observe that each negative term of the numerator is equal to a positive term into a power or product of terms $a$, $b$, $g$, contained in the denominator: this is the condition above referred to. The expansion thus consists only of terms each with the coefficient $+1$; for instance, a part of the function is

$$\frac{s\,(1-abg)}{1-a\,.\,1-b\,.\,1-c\,.\,1-g\,.\,1-q\,.\,1-u}, =\frac{s}{1-c\,.\,1-q\,.\,1-u}\cdot\frac{1-abg}{1-a\,.\,1-b\,.\,1-g},$$

where the first factor is the entire series of terms $sc^\delta q^\epsilon u^\zeta$, and the second factor is the series of terms $a^\alpha b^\beta g^\gamma$ omitting only those terms which are divisible by $abg$: and in the product of the two factors the terms are all distinct, so that the coefficients are still each $=1$. The same thing is true for every other pair of numerator terms: and (since the terms arising from each such pair are distinct from each other) in the expansion of the entire function the coefficients are each $=+1$. Hence (as in the case of the quartic) for any given deg-order, the terms in the expansion of the R.G.F. may be taken for the asyzygetic covariants of that deg-order; and if there are any other terms of the same deg-order, each of these must be a linear function, with numerical coefficients, of these asyzygetic covariants: thus deg-order 6.11, the expansion contains only the terms $a^2h$, $acd$, $be^2$; there is besides a term of the same deg-order, $ef$, which is not a term of the expansion, and hence $ef$ must be a linear function of $a^2h$, $acd$, $bc^2$; we in fact have $ef=a^2h-6acd-4bc^2$.

The terms in the expansion of the R.G.F. may be called "segregates," and the terms not in the expansion "congregates;" the theorem thus is: every congregate is a linear function, with determinate numerical coefficients, of the segregates of the same deg-order.

369. I stop to remark that the numerator of the R.G.F. may be written in the more compendious form

$$\begin{aligned}&(1-b^5)(1-v)+(1-b^3)(o+t)+(1-b^2)(e+k)+(1-b)f\\&+(1-ag^2)(d+h+j+m+dj+hj+j^2+jm)\\&+(1-bg)(l+jo+js)\\&+(1-b^2g)(i+n+p+jk)\\&+(1-abg)s\\&+(1-g)jt\\&+(1-a)w\,;\end{aligned}$$

but the first-mentioned form is, I think, the more convenient one.

370. It is to be noticed that the positive terms of the numerator are unity, the seventeen covariants $d, e, f, h, i, j, k, l, m, n, o, p, r, s, t, v, w$, and the products $j$ into $(d, h, j, k, m, o, s, t)$ where $j^2$ is reckoned as a product; in all, 26 terms. Disregarding the negative terms of the numerator the expansion would consist of these 26 terms, each multiplied by every combination whatever $a^\alpha b^\beta c^\gamma g^\delta q^\epsilon u^\zeta$ of the denominator terms $a, b, c, g, q, u$ (which for this reason might be called "reiterative"): the effect of the negative terms of the numerator is to remove from the expansion certain of the terms in question, thereby diminishing the number of the segregates: thus as regards the terms belonging to unity, any one of these which contains the factor $b^5$ is not a segregate but a congregate: and so as regards the terms belonging to $d$, any one of these which contains the factor $ag^2$ is a congregate: and the like in other cases.

For a given deg-order we have a certain number of segregates and a certain number of congregates: and the number of independent syzygies of that deg-order is precisely equal to the number of congregates: viz., each such syzygy may be regarded as giving a congregate in terms of the segregates: we have on the left hand side a congregate, or, to avoid fractions, a numerical multiple of the congregate, and on the right hand side a linear function, with numerical coefficients, of the segregates.

371. The syzygy is irreducible or reducible; and in the latter case it is, or is not, simply divisible: viz., if the congregate on the left hand side contains any congregate factor (the other factor being literal) then the syzygy is reducible: it is, in fact, obtainable from the syzygy (of a lower deg-order) which gives the value of such congregate factor. But there are here two cases; multiplying the lower syzygy by the proper factor, the right hand side may still contain segregates only, and then no further step is required: the original syzygy is nothing else than this lower syzygy, each side multiplied by the factor in question, and it is accordingly said to be simply divisible (S.D.). But contrariwise, the right hand side, as multiplied, may contain congregates which have to be replaced by their values in terms of the segregates of the same deg-order: the resulting expression is then no longer explicitly divisible by the introduced factor: and the original syzygy, although arising as above from a lower syzygy, is not this lower syzygy each side multiplied by a factor: viz., it is in this case not simply divisible.

For example (see the subsequent Table No. 96, under the indicated deg-orders) (6.6), from the syzygy $9d^2=aj-b^3+2bh-cg$, we deduce (7.11) the syzygy $9ad^2=a^2j-ab^3+2abh-acg$, which (all the terms on the right hand being segregates) requires no further reduction: it is a reducible and simply divisible syzygy. But we have (6.8) a syzygy giving $de$, and also (6.10) a syzygy giving $e^2$; multiplying the former of these by $e$ or the latter of them by $d$, we obtain a value of $de^2$, but in each case the right hand side contain terms which are not segregates, and have thus to be further reduced; the final formula (9.13) is

$$3de^2=-4a^2bj+3a^2dg+4ab^4-8ab^2h+4abcg-12b^2cd,$$

which is not divisible by any factor: the syzygy is thus reducible, but not simply divisible.

A syzygy which is not in the sense explained reducible, is said to be irreducible.

372. The number of irreducible syzygies is obviously finite: it has, however, the large value 179 as appears from the annexed diagram, showing the congregates determined by these several syzygies, and the deg-orders of the syzygies:—

| | | 1 | *d* | *e* | *f* | *h* | *i* | *j* | *k* | *l* | *m* | *n* | *o* | *p* | *r* | *s* | *t* | *v* | *w* | $j^2$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 1 | $b^5$<br>10.10 | $ag^2$<br>12.8 | $b^2$<br>7.9 | $b$<br>5.11 | $ag^2$<br>13.9 | $b^2g$<br>12.10 | $ag^2$<br>14.6 | $b^3$<br>9.7 | $bg$<br>11.9 | $ag^2$<br>15.7 | $b^2g$<br>14.8 | $b^3$<br>13.7 | $b^2g$<br>15.9 | $b^2g$<br>16.6 | $abg$<br>16.10 | $b^3$<br>17.7 | $b^5$<br>23.11 | $a$<br>19.5 | | 1 |
| 3.3 | *d* | | 6.6 | 6.8 | 6.12 | 7.7 | 7.9 | $ag^2$<br>17.9 | 8.6 | 8.10 | 9.5 | 9.7 | 10.4 | 10.8 | 11.5 | 12.6 | 14.4 | 16.4 | 21.3 | 13.5 | *d* |
| 3.5 | *e* | | | 6.10 | 6.14 | 7.9 | 7.11 | 8.6 | 8.8 | 8.12 | 9.7 | 9.9 | 10.6 | 10.10 | 11.7 | 12.8 | 14.6 | 16.6 | 21.5 | | *e* |
| 3.9 | *f* | | | | 6.18 | 7.13 | 7.15 | 8.10 | 8.12 | 8.16 | 9.11 | 9.13 | 10.10 | 10.14 | 11.11 | 12.12 | 14.10 | 16.10 | 21.9 | | *f* |
| 4.4 | *h* | | | | | 8.8 | 8.10 | $ag^2$<br>18.10 | 9.7 | 9.11 | 10.6 | 10.8 | 11.5 | 11.9 | 12.6 | 13.7 | 15.5 | 17.5 | 22.4 | 14.6 | *h* |
| 4.6 | *i* | | | | | | 8.12 | 9.7 | 9.9 | 9.13 | 10.8 | 10.10 | 11.7 | 11.11 | 12.8 | 13.9 | 15.7 | 17.7 | 22.6 | | *i* |
| 5.1 | *j* | | | | | | | $ag^2$<br>19.7 | $b^2g$<br>18.8 | 10.8 | $ag^2$<br>20.8 | 11.5 | $bg$<br>18.4 | 12.6 | 13.3 | $bg$<br>20.6 | $g$<br>20.2 | 18.2 | 23.1 | 15.3 | *j* |
| 5.3 | *k* | | | | | | | | 10.6 | 10.10 | 11.5 | 11.7 | 12.4 | 12.8 | 13.5 | 14.6 | 16.4 | 18.4 | 23.3 | 15.5 | *k* |
| 5.7 | *l* | | | | | | | | | 10.14 | 11.9 | 11.11 | 12.8 | 12.12 | 13.9 | 14.10 | 16.8 | 18.8 | 23.7 | | *l* |
| 6.2 | *m* | | | | | | | | | | 12.4 | 12.6 | 13.3 | 13.7 | 14.4 | 15.5 | 17.3 | 19.3 | 24.2 | 16.4 | *m* |
| 6.4 | *n* | | | | | | | | | | | 12.8 | 13.5 | 13.9 | 14.6 | 15.7 | 17.5 | 19.5 | 24.4 | | *n* |
| 7.1 | *o* | | | | | | | | | | | | 14.2 | 14.6 | 15.3 | 16.4 | 18.2 | 20.2 | 25.1 | 17.3 | *o* |
| 7.5 | *p* | | | | | | | | | | | | | 14.10 | 15.7 | 16.8 | 18.6 | 20.6 | 25.5 | | *p* |
| 8.2 | *r* | | | | | | | | | | | | | | 16.4 | 17.5 | 19.3 | 21.3 | 26.2 | | *r* |
| 9.3 | *s* | | | | | | | | | | | | | | | 18.6 | 20.4 | 22.4 | 27.3 | 19.5 | *s* |
| 11.1 | *t* | | | | | | | | | | | | | | | | 22.2 | 24.2 | 29.1 | 21.3 | *t* |
| 13.1 | *v* | | | | | | | | | | | | | | | | | 26.2 | 31.1 | | *v* |
| 18.0 | *w* | | | | | | | | | | | | | | | | | | 36.0 | | *w* |

Each term inside this diagram is a deg-order indicating the congregate determined by an irreducible syzygy: viz., the congregate is the product of the outside covariants in the line and column containing the deg-order, and of the literal factor (if any) placed immediately above the deg-order. Thus, line *d* and column *i*, 7.9 indicates the congregate *di*, but, same line and column *j*, 17.9 indicates the congregate $dj.\,ag^2, =adg^2j$.

Observe as regards the foregoing diagram, that $dj^2$ is irreducible (since neither $dj$ nor $j^2$ is segregate) and similarly $j^2h$, $j^3$, &c., are irreducible: we have thus the last or $j^2$ column of the diagram.

The simply divisible syzygies are infinite in number, as are also the reducible syzygies not simply divisible. There is obviously no use in writing down a simply divisible syzygy; but as regards the reducible syzygies not simply divisible, these require a calculation, and it is proper to give them as far as they have been obtained.

373. The following Table, No. 96, replaces Tables 88 and 89 of my Ninth Memoir. The arrangement is according to deg-orders, and the table is complete up to the deg-order 8.40: it shows for each deg-order the segregate covariants, and also the congregate covariants (if any), and the syzygies which are the expressions of these in terms of the segregates. When there are only segregates these are given in the same horizontal line with the deg-order; for instance, $\left|\; 5.9 \;\right|\; ab^2, ah, cd,$ shows that for the deg-order 5.9 the only covariants are the segregates $ab^2$, $ah$, $cd$; but when there are also congregates, the segregates are arranged in the same horizontal line with the deg-order, and the congregates, each in its own horizontal line, together with its expression as a linear function of the segregates: thus $\left|\begin{array}{c|c|cc} 5.11 & * & ai & ce \\ \cline{3-4} & bf & -1 & +1 \end{array}\right.$, the segregates are $ai$, $ce$, and there is a congregate $bf$ which is a linear function of these, $=-ai+ce$. The table gives the irreducible syzygies and also the reducible syzygies which are not simply divisible, but the simply divisible syzygies are indicated each by a reference to the divided syzygy which occurs previously in the table.

374. Any syzygy might of course be directly verified by substituting for the several covariants contained therein their expressions in terms of the coefficients and facients of the quintic. But it is to be remarked that among the syzygies, or easily deducible from them, we have (6.18) the before-mentioned equation $f^2=-a^3d+a^2bc-4c^3$, and also a set of 17 syzygies, the left hand sides of which are the covariants $g, h \ldots u, v, w$, each multiplied by $a$ or $a^2$, and which lead ultimately to the standard expressions of these covariants respectively, viz., each covariant multiplied by a proper power of $a$ can be expressed as a rational and integral function of $a, b, c, d, e, f$, linear as regards $f$: supposing them thus expressed, a far more simple verification of any syzygy would consist in substituting therein for the several covariants their expressions in the standard form, reducing if necessary by the equation $f^2=-a^3d+a^2bc-4c^3$: but of course, as to the syzygies used for obtaining the standard forms, this is only a verification if the standard forms have been otherwise obtained, or are assumed to be correct.

The 17 syzygies above referred to are

| Deg-ord. | | |
|---|---|---|
| 6.10 | $a^2g=$ | $12abd+4b^2c+e^2$, |
| 6.14 | $a^2h=$ | $6acd+4bc^2+ef$, |
| 5.11 | $ai=$ | $-bf+ce$, |
| 6.6 | $aj=$ | $b^3-2bh+cg+9d^2$, |
| 6.8 | $ak=$ | $-2bi+3de$, |
| 6.12 | $al=$ | $2ci-3df$, |
| 7.7 | $am=$ | $-2b^2d-cj+3dh$, |
| 7.9 | $an=$ | $b^2e-6bl-2ck-fg$, |
| 8.6 | $ao=$ | $2bn+ej$, |
| 8.10 | $ap=$ | $-2cn-fj$, |
| 9.5 | $aq=$ | $-2b^2j+bdg-12dm+hj$, |
| 9.7 | $ar=$ | $b^2k+bp-co+hk$, |
| 10.8 | $as=$ | $3bdk+3dp+2im$, |
| 12.6 | $at=$ | $bjk+jp-2mn$, |
| 13.5 | $18au=$ | $2agq+b^2gj+6bmj-6dj^2-ghj+no$, |
| 14.6 | $3av=$ | $2b^3q-8b^2j^2-2b^2gm+6bdgj-12bm^2+3et$, |
| 19.5 | $18aw=$ | $3b^2gt+b^2qo-4bj^2o-bgmo+18bmt+3dgjo-18djt-3ght-6m^2o$, |

the last four of these being, however, beyond the limits of the table: the expressions of $g$, $h$, $i$ are here in the standard form: the standard forms of the other covariants $j$, $k$ . . . $u$, $v$, $w$, will be given further on.

Table No. 96 (Segregates, Congregates, and Syzygies).

| Deg ord. | Congs. | Segregates. |
|---|---|---|
| 1. 1 | | 1 |
| 3 | | |
| 5 | | $a$ |
| 2. 0 | | |
| 2 | | $b$ |
| 4 | | |
| 6 | | $c$ |
| 8 | | |
| 10 | | $a^2$ |
| 3. 1 | | |
| 3 | | $d$ |
| 5 | | $e$ |
| 7 | | $ab$ |
| 9 | | $f$ |
| 11 | | $ac$ |
| 13 | | |
| 15 | | $a^3$ |

Table No. 96 (continued).

| Deg-ord. | Congs. | Segregates. |
|---|---|---|
| 4. 0 | | $g$ |
| 2 | | |
| 4 | | $b^2$, $h$ |
| 6 | | $i$ |
| 8 | | $ad$, $bc$ |
| 10 | | $ae$ |
| 12 | | $a^2b$, $c^2$ |
| 14 | | $af$ |
| 16 | | $a^3c$ |
| 18 | | |
| 20 | | $a^5$ |
| 5. 1 | | $j$ |
| 3 | | $k$ |
| 5 | | $ag$, $bd$ |
| 7 | | $be$, $l$ |
| 9 | | $ab^2$, $ah$, $cd$ |
| 11 | * | $ai$, $ce$ |
| | $bf$ | $-1$ $+1$ |
| 13 | | $a^2d$, $abc$ |
| 15 | | $a^2e$, $cf$ |
| 17 | | $a^3b$, $ac^2$ |
| 19 | | $a^2f$ |
| 21 | | $a^3c$ |
| 23 | | |
| 25 | | $a^5$ |
| 6. 0 | | |
| 2 | | $bg$, $m$ |
| 4 | | $n$ |
| 6 | * | $aj$, $b^3$, $bh$, $cg$ |
| | $d^2.9$ | $+1$ $-1$ $+2$ $-1$ |
| 8 | * | $ak$, $bi$ |
| | $de.3$ | $+1$ $+2$ |
| 10 | * | $a^2g$, $abd$, $b^2c$, $ch$ |
| | $e^2$ | $+1$ $-12$ $-4$ . |
| 12 | * | $abe$, $al$, $ci$ |
| | $df.3$ | . $-1$ $+2$ |
| 14 | * | $a^2b^2$, $a^2h$, $acd$, $bc^2$ |
| | $ef$ | . $+1$ $-6$ $-4$ |
| 16 | * | $a^2i$, $ace$ |
| | $abf$ | . . . . . . . . . . . . . . . . . . . . . . S.D. 5.11, $bf$ |

Table No. 96 (continued).

| Deg-ord. | Congs. | Segregates. | | | | | |
|---|---|---|---|---|---|---|---|
| 6. 18 | * | $a^3d$, | $a^2bc$, | $c^3$ | | | |
| | $f^2$ | −1 | +1 | −4 | | | |
| 20 | | $a^3e$, | $acf$ | | | | |
| 22 | | $a^4b$, | $a^2c^2$ | | | | |
| 24 | | $a^3f$ | | | | | |
| 26 | | $a^4c$ | | | | | |
| 28 | | | | | | | |
| 30 | | $a^6$ | | | | | |
| 7. 1 | | $o$ | | | | | |
| 3 | | $bj$, | $dg$ | | | | |
| 5 | | $bk$, | $eg$, | $p$ | | | |
| 7 | * | $abg$, | $am$, | $b^2d$, | $cj$ | | |
| | $dh$.3 | 0 | +1 | +2 | +1 | | |
| 9 | * | $an$, | $bl$, | $ck$, | $fg$ | | |
| | $b^2e$ | +1 | +6 | +2 | +1 | | |
| | $di$.3 | 0 | +1 | +1 | | | |
| | $eh$ | 0 | +4 | +2 | +1 | | |
| 11 | * | $a^2j$, | $ab^3$, | $abh$, | $acg$, | $bcd$ | |
| | $ad^2$ | . | . | . | . | . | S.D. 6.6, $d^2$ |
| | $ei$ | . | . | −1 | +1 | −6 | |
| 13 | * | $a^2k$, | $abi$, | $bce$, | $cl$ | | |
| | $ade$ | . | . | . | . | | S.D. 6.8, $de$ |
| | $b^2f$ | . | . | . | . | | S.D. 5.11, $bf$ |
| | $fh$.3 | −1 | −2 | +3 | −6 | | |
| 15 | * | $a^3g$, | $a^2bd$, | $ab^2c$, | $ach$, | $c^2d$ | |
| | $ae^2$ | . | . | . | . | . | S.D. 6.10, $e^2$ |
| | $fi$ | . | +1 | −1 | +1 | −6 | |
| 17 | * | $a^2be$, | $a^2l$, | $aci$ | $c^2e$ | | |
| | $adf$ | . | . | . | . | | S D. 6.12, $df$ |
| | $bcf$ | . | . | . | . | | S.D. 5.11, $bf$ |
| 19 | * | $a^3b^2$, | $a^3h$, | $a^2cd$, | $abc^2$ | | |
| | $aef$ | . | . | . | . | | S.D. 6.14, $ef$ |
| 21 | * | $a^3i$, | $a^2ce$, | $c^2f$ | | | |
| | $a^2bf$ | . | . | . | | | S.D. 5.11, $bf$ |
| 23 | * | $a^4d$, | $a^3bc$, | $ac^3$ | | | |
| | $af^2$ | . | . | . | | | S.D., 6.18, $f^2$ |

Table No. 96 (continued).

| Deg-ord. | Congs. | Segregates. | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 25 | | $a^4e$, | $a^2cf$ | | | | | | |
| 27 | | $a^5b$, | $a^3c^2$ | | | | | | |
| 29 | | $a^4f$ | | | | | | | |
| 31 | | $a^5c$ | | | | | | | |
| 33 | | | | | | | | | |
| 35 | | $a^7$ | | | | | | | |
| 8. 0 | | $g^2$, | $q$ | | | | | | |
| 2 | | $r$ | | | | | | | |
| 4 | | $b^2g$, | $bm$, | $dj$, | $gh$ | | | | |
| 6 | * | $ao$, | $bn$, | $gi$ | | | | | |
| | $dk.3$ | +1 | −3 | −1 | | | | | |
| | $ej$ | +1 | −2 | | | | | | |
| 8 | * | $abj$, | $adg$, | $b^4$, | $b^2h$, | $bcg$, | $cm$ | | |
| | $bd^2$ | | | | | | | | S.D. 6.6, $d^2$ |
| | $ek$ | −4 | +3 | +4 | −6 | +2 | | | |
| | $h^2.3$ | +4 | −3 | −4 | +8 | −1 | +12 | | |
| 10 | * | $abk$, | $aeg$, | $ap$, | $b^2i$, | $cn$ | | | |
| | $bde$ | | | | | | | | S.D. 6.8, $de$ |
| | $dl.9$ | +2 | . | +3 | +1 | +3 | | | |
| | $fj$ | . | . | −1 | . | −2 | | | |
| | $hi.3$ | +1 | . | . | +2 | −3 | | | |
| 12 | * | $a^2bg$, | $a^2m$, | $ab^2d$, | $acj$, | $b^3c$, | $bch$, | $c^2g$ | |
| | $adh$ | | | | | | | | S.D. 7.7, $dh$ |
| | $be^2$ | | | | | | | | S.D. 6.10, $e^2$ |
| | $cd^2$ | | | | | | | | S.D. 6.6, $d^2$ |
| | $el$ | . | −1 | −2 | +1 | −2 | +2 | | |
| | $fk$ | . | +1 | . | −1 | +4 | −6 | +2 | |
| | $i^2$ | . | . | −1 | . | +1 | −2 | +1 | |
| 14 | * | $a^2n$, | $abl$, | $ack$, | $afg$, | $bci$ | | | |
| | $ab^2e$ | | | | | | | | S.D. 7.9, $b^2e$ |
| | $adi$ | | | | | | | | S.D. 7.9, $di$ |
| | $aeh$ | | | | | | | | S.D. 7.9, $eh$ |
| | $bdf$ | | | | | | | | S.D. 6.12, $df$ |
| | $cde$ | | | | | | | | S.D. 6.8, $de$ |
| 16 | * | $a^3j$, | $a^2b^3$, | $a^2bh$, | $a^2cg$, | $abcd$, | $b^2c^2$, | $c^2h$ | |
| | $a^2d^2$ | | | | | | | | S.D. 6.6, $d^2$ |
| | $aei$ | | | | | | | | S.D. 7.11, $ei$ |
| | $bef$ | | | | | | | | S D. 6.14, $ef$ |
| | $ce^2$ | | | | | | | | S.D. 6.10, $e^2$ |
| | $fl.3$ | +1 | −1 | +2 | −1 | −3 | −6 | +6 | |

Table No. 96 (continued).

| Deg-ord. | Congs. | Segregates. | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 8. 18 | * | $a^3k$, | $a^2bi$, | $abce$, | $acl$, | $c^2i$ | | | |
| | $a^2de$ | | | | | | | | S.D. 6.8, $de$ |
| | $ab^2f$ | | | | | | | | S.D. 5.11, $bf$ |
| | $afh$ | | | | | | | | S.D. 7.13, $fh$ |
| | $cdf$ | | | | | | | | S.D. 6.12, $df$ |
| 20 | * | $a^4g$, | $a^3bd$, | $a^2b^2c$, | $a^2ch$, | $ac^2d$, | $bc^3$ | | |
| | $a^2e^2$ | | | | | | | | S.D. 6.10, $e^2$ |
| | $afi$ | | | | | | | | S.D. 7.15, $fi$ |
| | $bf^2$ | | | | | | | | S.D. 6.18, $f^2$ |
| | $cef$ | | | | | | | | S.D. 6.14, $ef$ |
| 22 | * | $a^3be$, | $a^3l$, | $a^2ci$, | $abcf$, | $ac^2e$ | | | |
| | $a^2df$ | | | | | | | | S.D. 6.12, $df$ |
| 24 | * | $a^4b^2$, | $a^4h$, | $a^3cd$, | $a^2b^2c^2$, | $c^4$ | | | |
| | $a^2ef$ | | | | | | | | S.D. 6.14, $ef$ |
| | $cf^2$ | | | | | | | | S.D. 6.18, $f^2$ |
| 26 | * | $a^4i$, | $a^3ce$, | $ac^2f$ | | | | | |
| | $a^3bf$ | | | | | | | | S.D. 5.11, $bf$ |
| 28 | * | $a^5d$, | $a^4bc$, | $a^2c^3$ | | | | | |
| | $a^2f^2$ | | | | | | | | S.D. 6.18, $f^2$ |
| 30 | | $a^5e$, | $a^3cf$ | | | | | | |
| 32 | | $a^6b$, | $a^4c^2$ | | | | | | |
| 34 | | $a^5f$ | | | | | | | |
| 36 | | $a^6c$ | | | | | | | |
| 38 | | | | | | | | | |
| 40 | | $a^8$ | | | | | | | |
| 9. 1 | | $gj$ | | | | | | | |
| 3 | | $bo$, | $gk$, | $s$ | | | | | |
| 5 | * | $ag^2$, | $aq$, | $b^2j$, | $bdg$, | $hj$ | | | |
| | $dm$.12 | . | −1 | −2 | +1 | +1 | | | |
| 7 | * | $ar$, | $beg$, | $bp$, | $co$, | $gl$ | | | |
| | $b^2k$.3 | +1 | . | −5 | −1 | +1 | | | |
| | $dn$.3 | −1 | . | −1 | −1 | | | | |
| | $em$.3 | +2 | . | +2 | +1 | −1 | | | |
| | $hk$.3 | +2 | . | +2 | +4 | −1 | | | |
| | $ij$ | . | . | +1 | +1 | | | | |
| 9 | * | $ab^2g$, | $abm$, | $adj$, | $agh$, | $b^3d$, | $bcj$, | $cdg$ | |
| | $bdh$.3 | . | +1 | . | . | +2 | +1 | | |
| | $d^3$.27 | . | +2 | +3 | . | +1 | +2 | −3 | |
| | $en$ | +1 | +6 | . | −1 | . | +2 | | |
| | $ik$ | . | −1 | . | . | . | −3 | +3 | |

## Table No. 96 (continued).

| Deg-ord. | Congs. | Segregates. | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 9. 11 | * | $a^2o$, | $abn$, | $agi$, | $b^2l$, | $bck$, | $ceg$, | $cp$ | | | | |
| | $adk$ | | | | | | | | | | | S.D. 8.6, $dk$ |
| | $aej$ | | | | | | | | | | | S.D. 8.6, $ej$ |
| | $b^3e$ | . | +1 | −1 | +6 | +2 | +1 | | | | | |
| | $bdi$ | | | | | | | | | | | S.D. 7.9, $bi$ |
| | $beh$ | . | . | −1 | +4 | +2 | +1 | | | | | |
| | $bfg$ | | | | | | | | | | | S.D. 5.11, $bf$ |
| | $d^2e$.9 | +1 | −3 | −1 | +2 | +2 | | | | | | |
| | $fm$.3 | +1 | +3 | +1 | . | −3 | . | −3 | | | | |
| | $hl$.3 | +1 | −3 | −1 | +2 | +5 | . | −6 | | | | |
| 13 | * | $a^2bj$, | $a^2dg$, | $ab^4$, | $ab^2h$, | $abcg$, | $acm$, | $b^2cd$, | $c^2j$ | | | |
| | $abd^2$ | | | | | | | | | | | S.D. 6.6, $d^2$ |
| | $aek$ | | | | | | | | | | | S.D. 8.8, $ek$ |
| | $ah^2$ | | | | | | | | | | | S.D. 8.8, $h^2$ |
| | $bei$ | | | | | | | | | | | S.D. 7.11, $ei$ |
| | $cdh$ | | | | | | | | | | | S.D. 7.7, $dh$ |
| | $de^2$.3 | −4 | +3 | +4 | −8 | +4 | . | −12 | | | | |
| | $fn$ | −2 | +1 | +2 | −3 | +1 | −2 | . | +2 | | | |
| | $il$ | −1 | . | +1 | −2 | +1 | −3 | − 3 | +3 | | | |
| 10. 0 | | | | | | | | | | | | |
| 2 | | $bg^2$, | $bq$, | $gm$, | $j^2$ | | | | | | | |
| 4 | * | $br$, | $gn$, | $jk$ | | | | | | | | |
| | $do$.3 | +2 | | −1 | | | | | | | | |
| 6 | * | $agj$, | $b^3g$, | $b^2m$, | $bdj$, | $bgh$, | $cg^2$, | $cq$ | | | | |
| | $d^2g$ | | | | | | | | | | | S.D. 6.6, $d^2$ |
| | $eo$ | +1 | +2 | +12 | −12 | −2 | | | | | | |
| | $hm$.3 | −1 | +1 | + 6 | +12 | −2 | +1 | −3 | | | | |
| | $k^2$ | +1 | −1 | − 4 | −12 | +2 | −1 | | | | | |
| 8 | * | $abo$, | $agk$, | $as$, | $b^2n$, | $bgi$, | $cr$ | | | | | |
| | $bdk$ | | | | | | | | | | | S.D. 8.6, $dk$ |
| | $bej$ | | | | | | | | | | | S.D. 8.6, $ej$ |
| | $deg$ | | | | | | | | | | | S.D. 6.8, $de$ |
| | $dp$.9 | −5 | . | +3 | +15 | +5 | − 6 | | | | | |
| | $hn$.3 | −6 | +1 | . | +18 | +5 | −12 | | | | | |
| | $im$.3 | +1 | . | . | − 3 | −1 | + 3 | | | | | |
| | $jl$.3 | −5 | . | +3 | +15 | +5 | −12 | | | | | |
| 10 | * | $a^2g^2$, | $a^2q$, | $ab^2j$, | $abdg$, | $ahj$, | $b^3h$, | $b^2cg$, | $bcm$, | $cdj$, | $cgh$. | |
| | $adm$ | | | | | | | | | | | S.D. 9.5, $dm$ |
| | $b^5$.8 | . | +1 | +10 | −9 | +3 | +12 | . | +32 | −12 | − 4 | |
| | $b^2d^2$.72 | . | −1 | − 2 | +9 | −3 | + 4 | −8 | −32 | +12 | + 4 | |
| | $bek$.2 | . | +1 | + 2 | −3 | +3 | . | +4 | +32 | −12 | − 4 | |
| | $bh^2$.6 | . | −1 | − 2 | +3 | −3 | + 4 | −2 | − 8 | +12 | − 4 | |
| | $d^2h$.27 | . | −1 | − 2 | +3 | . | + 1 | −2 | − 8 | +12 | + 1 | |
| | $e^2g$ | | | | | | | | | | | S.D. 6.10, $e^2$ |
| | $ep$ | . | . | . | . | −1 | . | −2 | −12 | + 6 | + 2 | |
| | $fo$.2 | . | +1 | + 2 | −3 | +7 | . | +4 | +24 | −24 | − 4 | |
| | $in$.4 | . | −1 | − 2 | +3 | −5 | . | . | . | +12 | | |
| | $kl$.4 | . | +1 | + 2 | −1 | −1 | . | . | +16 | +12 | | |

Table No. 96 (concluded).

| Deg.-ord. | Congs. | Segregates. | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 11. 1 | | $go$, | $t$ | | | | | |
| 3 | | $bgj$, | $dg^2$, | $dq$, | $jm$ | | | |
| 5 | * | $b^2o$, | $bgk$, | $bs$, | $eg^2$, | $eq$, | $gp$ | |
| | $dr$.18 | −2 | + 5 | − 6 | . | −3 | +3 | |
| | $ho$.3 | −2 | +11 | −24 | . | −3 | +6 | |
| | $jn$.1 | +1 | − 3 | + 6 | . | +1 | −2 | |
| | $km$.6 | −2 | + 5 | −12 | . | −3 | +3 | |
| 12. 0 | | $g^3$, | $g^2q$, | $u$ | | | | |
| 2 | | $gr$, | $jo$ | | | | | |
| 4 | * | $b^2g^2$, | $b^2q$, | $bgm$, | $bj^2$, | $dgj$, | $g^2h$, | $hg$ |
| | $ko$ | . | −2 | −2 | −4 | +3 | | |
| | $m^2$.12 | . | +2 | +1 | +4 | −3 | . | −3 |
| 13. 1 | | $g^2j$, | $jq$, | $v$ | | | | |
| 3 | * | $bgo$, | $bt$, | $g^2k$, | $gs$, | $kg$ | | |
| | $jr$.2 | . | −2 | . | +1 | −1 | | |
| | $mo$.2 | . | −4 | . | +1 | −1 | | |
| 14. 0 | | | | | | | | |
| 2 | | $bg^3$, | $bgq$, | $bu$, | $g^2m$, | $gj^2$, | $mq$, | $o^2$ |
| 4 | * | $bgr$, | $bjo$, | $g^2n$, | $gjk$, | $js$, | $nq$ | |
| | $dgo$ | . . . | . . . | . . . | . . . | . . . | . . . | . . . . . . . . . . . S.D. 10.4, *do* |
| | $dt$.18 | +1 | +2 | . | −1 | +6 | +3 | |
| | $mr$.12 | +1 | +2 | . | −1 | . | +3 | |

*Theory of the Canonical Form.*—Article Nos. 375 to 381, and Tables Nos. 97 and 98.

375. As the small italic letters have been used to represent the covariants, different letters are required for the coefficients of the quintic, and (using also new letters for the facients) I take the quintic to be $(\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}, \mathrm{e}, \mathrm{f} \between \xi, \eta)^5$. Considering a linear transformation of $\frac{1}{\mathrm{a}}(\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}, \mathrm{e}, \mathrm{f} \between \xi, \eta)^5$, viz.

$$\frac{1}{\mathrm{a}}(\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}, \mathrm{e}, \mathrm{f} \between \xi - \mathrm{b}\eta, \mathrm{a}\eta)^5$$

this is

$$= \begin{array}{rcccccc} & \xi^5 & 5\xi^4\eta & 10\xi^3\eta^2 & 10\xi^2\eta^3 & 5\xi\eta^4 & \eta^5 \\ \hline & 1 & -b & +\ b^2 & -\ b^3 & +\ b^4 & -\ \ b^5) \\ +b\ ( & & 1 & -2b & +3b^2 & -4b^3 & +5\ b^4) \\ +ac\ ( & & & 1 & -3b & +6b^2 & -10b^3) \\ +a^2d\ ( & & & & 1 & -4b & +10b^2) \\ +a^3e\ ( & & & & & 1 & -\ 5b) \\ +a^4f\ ( & & & & & & 1) \end{array}$$

which is

$$= (\begin{array}{|c|c|ll|ll|ll|ll|} \hline 1 & 0 & ac & +1 & a^2d & +1 & a^3e & +1 & a^4f & +\ 1 \\ & & b^2 & -1 & abc & -3 & a^2bd & -4 & a^3be & -\ 5 \\ & & & & b^3 & +2 & ab^2c & +6 & a^2b^2d & +10 \\ & & & & & & b^4 & -3 & ab^3c & -10 \\ & & & & & & & & b^5 & +\ 4 \\ \hline \end{array} \between \xi,\ \eta)^5$$

The values of $a$, $b$, $c$, $d$, $e$, $f$ considered for a moment as denoting the leading coefficients of the several covariants ultimately represented by these letters respectively, are

| $a$ | | $b$ | | $c$ | | $d$ | | $e$ | | $f$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| a | +1 | ae | +1 | ac | +1 | ace | +1 | $a^2f$ | + 1 | $a^2d$ | +1 |
| | | bd | −4 | $b^2$ | −1 | $ad^2$ | −1 | abe | + 5 | abc | −3 |
| | | $c^2$ | +3 | | | $b^2e$ | −1 | acd | + 2 | $b^3$ | −2 |
| | | | | | | bcd | +2 | $b^2d$ | + 8 | | |
| | | | | | | $c^3$ | −1 | $bc^2$ | −10 | | |

satisfying, as they should do, the relation $f^2=-a^3d+a^2bc-4c^3$.

Hence forming the values of $a^2b-3c^2$ and $a^2e-2cf$, it appears that the value of the last-mentioned quintic function is

$$(1,\ 0,\ c,\ f,\ a^2b-3c^2,\ a^2e-2cf \between \xi,\ \eta)^5.$$

Writing herein $x$, $y$ in place of $\xi$, $\eta$, and now using $a$, $b$, $c$, $d$, $e$, $f$ to denote, not the leading coefficients but the covariants themselves ($a$ the original quintic, with $\xi$, $\eta$ as facients), we have the form

$$\mathrm{A},\ =(1,\ 0,\ c,\ f,\ a^2b-3c^2,\ a^2e-2cf \between x,\ y)^5,$$

a new quintic, which is the canonical form in question: the covariants hereof (reckoning the quintic itself as a covariant) will be written A, B, C . . . V, W; and will be spoken of as capital covariants.

376. The fundamental property is: Every capital covariant, say I, has for its leading coefficient the corresponding covariant $i$ multiplied into a power of $a$: and this follows as an immediate consequence of the foregoing genesis of A. The covariant $i$ of the form $\frac{1}{\mathrm{a}}(\mathrm{a, b, c, d, e, f} \between \xi, \eta)^5$ has a leading coefficient $=\frac{1}{\mathrm{a}^4}(\mathrm{a^2cf - a^2de} + \&\mathrm{c.})$ which, when $a, b, c, d, e, f \ldots i$ denote leading coefficients, is $=i$ into a power of a: and upon substituting for the quintic the linear transformation thereof

$$(1,\ 0,\ c, f,\ a^2b-3c^2,\ a^2e-2cf \between \xi, \eta)^5,$$

(observing that in the transformation $\xi, \eta$ into $\xi - \mathrm{b}\eta$, $\mathrm{a}\eta$ the determinant of substitution is $=\mathrm{a}$), the value is still $=i$ into a power of a; or using the relation $a=\mathrm{a}$, say the value is $=i$ into a power of $a$. Now the covariant $i$ is the same function of the covariants $a, b, c, d, e, f$ that the leading coefficient $i$ is of the leading coefficients $a, b, c, d, e, f$; hence the italic letters now denoting covariants the leading coefficient still is $=i$ into a power of $a$: which is the above-mentioned theorem.

377. To show how the transformation is carried out, consider, for example, the covariant B: this is obtained from the corresponding covariant of $(\mathrm{a, b, c, d, e, f} \between \xi, \eta)^5$ that is

$$\left(\begin{array}{ll|ll|ll} \mathrm{ae} & 1 & \mathrm{af} & 1 & \mathrm{bf} & 1 \\ \mathrm{bd} & -4 & \mathrm{be} & -3 & \mathrm{ce} & -4 \\ \mathrm{c}^2 & +1 & \mathrm{cd} & +1 & \mathrm{d}^2 & +3 \end{array} \between \xi, \eta\right)^2.$$

by changing the variables, and for the coefficients

$$\begin{array}{lcccccc} & \mathrm{a}, & \mathrm{b}, & \mathrm{c}, & \mathrm{d}, & \mathrm{e}, & \mathrm{f} \\ \text{writing} & 1, & 0, & c, & f, & a^2b-3c^2, & a^2e-2cf; \end{array}$$

thus the coefficients are

$$\begin{array}{lll} \text{First.} & \text{Second.} & \text{Third.} \\ 1(a^2b-3c^2) & 1(a^2e-2cf) & \phantom{=}\ -4c(a^2b-3c^2) \\ +3c^2 & +2cf & \phantom{=}\ +3f^2 \\ =a^2b & =a^2e & =\ -4a^2bc+12c^3 \\ & & \phantom{=}\ +3(-a^3d+a^2bc-4c^3) \\ & & =\ a^2(-3ad-bc) \end{array}$$

and we have thus the expression of B (see the Table No. 97); and similarly for the other capital covariants C, D ... V, W: in every case the coefficients are obtained in the standard form; that is, as rational and integral functions of $a, b, c, d, e, f$, linear as regards $f$.

378. It will be observed that there is in each case a certain power of $a$ which explicitly divides all the coefficients and is consequently written as an exterior factor:

disregarding these exterior factors, the leading coefficients for B, C, D, E, F are $b$, $c$, $ad$, $e$, $f$ respectively; that for G is $12abd+4b^2c+e^2$, which must be $=g$ into a power of $a$, and (in Table 97) is given as $=a^2g$, similarly that for H is $6acd+4bc^2+ef$, which must be $=h$ into a power of $a$, and is given as $=a^2h$ : and so in the other cases. The index of $a$ is at once obtained by means of the deg-order, which is in each case inserted at the foot of the coefficient.

For A, B, C, E, F there is no power of $a$ as an interior factor, and for the invariants G, Q, U we may imagine the interior factor thrown together with the exterior factor, ($\mathrm{G}=a^6g$, &c.): whence disregarding the exterior factors, we may say that for A, B, C, E, F, G, Q, U the standard forms are also "divided" forms. But take any other covariant—for instance, D : the leading coefficient is $ad$, having the interior factor $a$; and this being so it is found that all the following coefficients will divide by $a$ (the quotients being of course expressible only in terms of the covariants subsequent to $f$): thus the second coefficient of D is $-bf+ce$, and (5.11) we have $-bf+ce=ai$, or the coefficient divided by $a$ is $=i$; and so for the other coefficients of D; or throwing out the factor $a$, we obtain for D an expression of the form $(d, i, \ldots \between x, y)^3$, see the Table 98 : this is the "divided" form of D : and we have similarly a divided form for every other capital covariant. All that has been required is that each coefficient of the divided form shall be expressed as a rational and integral function of the covariants $a, b, c \ldots v, w$ : and the form is not hereby made definite : to render it so the coefficient must be expressed in the segregate form. But there is frequently the disadvantage that we thus introduce fractions; for instance, the last coefficient of D is $=-ci+df$, where to get rid of the congregate term $df$ we have (6.12), $3df=-al+2ci$, and the segregate form of the coefficient is $=-\tfrac{1}{3}al+\tfrac{2}{3}ci$.

379. We have in regard to the Canonical form a differential operator which is analogous to the two differential operators $xd_y-\{xd_y\}$, $yd_x-\{yd_x\}$ considered in the Introductory Memoir (1854). Let $\delta$ denote a differentiation in regard to the constants under the conditions

$$\begin{aligned}
\delta a &= 0,\\
\delta b &= e,\\
\delta c &= 3f,\\
\delta d &= \frac{1}{a}(-bf+ce),\ (=i),\\
\delta e &= -6ad-10bc,\\
\delta f &= 2a^2b-18c^2,
\end{aligned}$$

which (as at once verified) are consistent with the fundamental relation

$$f^2=-a^3d+a^2bc-4c^3\,;$$

then it is easy to verify that

$$\left(x\frac{d}{dy}-4cy\frac{d}{dx}-\delta\right)\mathrm{A}=0\,;$$

and this being so, any other covariant whatever, expressed in the like standard form, is reduced to zero by the operator

$$x\frac{d}{dy}-4cy\frac{d}{dx}-\delta\,;$$

and we have thus the means of calculating the covariant when the leading coefficient is known.

Thus, considering the covariant B, the expression of which has just been obtained, $=(\mathrm{B}_0, \mathrm{B}_1, \mathrm{B}_2 \between x, y)^2$, suppose : the equation to be satisfied is

$$\begin{array}{l} x\,(\mathrm{B}_1x+2\mathrm{B}_2y \qquad\quad) \\ -4cy\,(\qquad\quad 2\mathrm{B}_0x+\mathrm{B}_1y) \\ -x^2\delta\mathrm{B}_0\ -xy\delta\mathrm{B}_1-y^2\delta\mathrm{B}_2\ =0, \end{array}$$

viz., we have

$$\begin{array}{r} \mathrm{B}_1 \qquad -\delta\mathrm{B}_0=0, \\ 2\mathrm{B}_2-\delta c\mathrm{B}_0-\delta\mathrm{B}_1=0, \\ -4c\mathrm{B}_1-\delta\mathrm{B}_2=0\,; \end{array}$$

which (omitting, as we may do, the outside factor $a^2$) are satisfied by the foregoing values $\mathrm{B}_0, \mathrm{B}_1, \mathrm{B}_2=b, e, -3ad-bc$. And if we assume only $\mathrm{B}_0=b$, then the first equation gives at once the value $\mathrm{B}_1=e$, the second equation then gives $\mathrm{B}_2=-3ad-3bc$; and the third equation is satisfied identically, viz., the equation is

$$-4ce+\delta(3ad+bc)=0,$$

that is

$$\begin{array}{lll} -4ce & = \ -4ce & =0. \\ +c\delta b & \ \ +\ c\ .\ e & \\ +b\delta e & \ \ +\ b\ .\ 3f & \\ +3a\delta d & \ \ +3(-bf+ce) & \end{array}$$

which is right.

Of course every invariant must be reduced to zero by the operation $\delta$ : thus we have Table No. 97,

$$\begin{array}{rl} a^2g= & 12\ abd \\ & +\ 4\ b^2c \\ & +\ 1\ e^2, \end{array}$$

and thence

$$\begin{array}{llllllll} & & & & & ade & b^2f & bce \\ \hline a^2\delta g= & (12ad+8bc) & \delta b, & = (12ad+8bc) & e, & =+12 & & +\ 8 \\ & +4b^2 & .\ \delta c & +4b^2 & .\ 3f & & +12 & \\ & +12ab & .\ \delta d & +12b(-bf+ce) & & & -12 & +12 \\ & +2e & .\ \delta e & +2e(-6ad-10bc) & & -12 & & -20 \end{array}$$

which is $=0$, as it should be.

380. As already remarked, the leading coefficients of H, I, J, &c., are each of them equal to a power of $a$ into the corresponding covariant $h, i, j \ldots$; hence, supposing these leading coefficients, or, what is the same thing, the standard expressions of the covariants $h, i, j \ldots v, w$ to be known, we can calculate the values of $\delta h, \delta i, \delta j, \ldots \delta v, \delta w$ $(=0$, since $w$ is an invariant$)$: and the operation $\delta$ instead of being applicable only to the forms containing $a, b, c, d, e, f$, becomes applicable to forms containing any of the covariants. The values of $\delta a, \delta b, \ldots \delta v, \delta w$ can, it is clear, be expressed in terms of segregates; and this is obviously the proper form: but for $\delta r$, $\delta t$, and $\delta v$, for which the segregate forms are fractional, I have given also forms with integer coefficients. The entire series is

| Deg-order. | |
|---|---|
| 2.8 | $\delta a = 0,$ |
| 3.5 | $\delta b = e,$ |
| 3.9 | $\delta c = 3f,$ |
| 4.6 | $\delta d = i,$ |
| 4.8 | $\delta e = -6ad-10bc,$ |
| 4.12 | $\delta f = 2a^2b-18c^2,$ |
| 5.3 | $\delta g = 0,$ |
| 5.7 | $\delta h = 2be-4l,$ |
| 5.9 | $\delta i = -2ab^2+2ah-18cd,$ |
| 6.4 | $\delta j = -n,$ |
| 6.6 | $\delta k = -2aj+6b^3-9bh+3cg,$ |
| 6.10 | $\delta l = -3abd-7b^2c+7ch,$ |
| 7.5 | $\delta m = -bk-p,$ |
| 7.7 | $\delta n = 4cj,$ |
| 8.4 | $\delta o = b^2g+6bm-6dj-gh,$ |
| 8.8 | $\delta p = 8abj-5adg-10b^4+15b^2h-5bcg+10cm,$ |
| 9.3 | $\delta q = 0,$ |
| 9.5 | $\delta r = \frac{1}{2}(aq+6b^2j-5bdg-jh), \quad =2b^2j-2bdg-6dm,$ |
| 10.6 | $\delta s = -2agj+2b^3g+3b^2m+21bdj-4bgh+2cg^2-3cq,$ |
| 12.4 | $\delta t = \frac{1}{2}(bgm+4bj^2-3dgj-hq), \quad =-b^2q+hq+6m^2,$ |
| 13.3 | $\delta u = 0,$ |
| 14.4 | $\delta v = \frac{1}{6}(-5bgr-10bjo+5gjk-12js-9nq), \quad =-6dt-6mr+nq.$ |
| 19.3 | $\delta w = 0.$ |

It is obvious that for every covariant whatever written in the denumerate form $(\mathrm{I}_0, \mathrm{I}_1 \ldots \between x, y)^a$, the second coefficient is equal to the first coefficient operated upon by $\delta$; so that the foregoing formulæ give, in fact, the second coefficients of the several covariants.

381. It is worth noticing how very much the formulæ of Table No. 97 simplify

themselves, if one of the covariants $b$, $c$, $d$, $e$ vanishes, in particular if $b$ vanishes. Suppose $b=0$; writing also (although this makes but little difference) $a=1$, we have

$$\begin{aligned}
a &= 1,\\
b &= 0,\\
c &= c,\\
d &= d,\\
e &= e,\\
f^2 &= -d-4c^3,\\
g &= e^2,\\
h &= 6cd+ef,\\
i &= ce,\\
j &= 9d^2+ce^2,\\
k &= 3de,\\
l &= -3df+2c^2e,\\
m &= 9cd^2+3def-c^2e^2,\\
n &= -6cde-e^2f,\\
o &= 9d^2e+ce^3,\\
p &= -9d^2f+12c^2de+ce^2f,\\
q &= -54cd^3-27d^2ef+18c^2de^2+ce^3f,\\
r &= 9cd^2e+3de^2f-c^2e^3,\\
s &= -27d^3f+54c^2d^2e+9cde^2f-2c^3e^3,\\
t &= -81d^4f-6d^2e^3+216c^2d^3e+54cd^2e^2f-24c^3de^3-c^2e^4f,\\
u &= -27d^5-18cd^3e^2-4d^2e^3f+c^2de^4,\\
v &= -81d^4ef-6d^2e^4+216c^2d^3e^2+54cd^2e^3f-24c^3de^4-1c^2e^5f,\\
w &\ \text{(not calculated).}
\end{aligned}$$

These values are very convenient for the verification of syzygies, &c.: take, for instance, the before-mentioned relation $\delta v=-6dt-6mr+nq$, that is, if $V=(V_0, V_1 \between x, y)$, then $V_1=-6dt-6mr+nq$: calculating the three products on the right hand side, observing that $f^2$ when it occurs is to be replaced by its value $-d-4c^3$, and taking their sum, the figures are as follows:—

| | $-6dt$ | $-6mr$ | $+nq$ | *Sum* |
|---|---|---|---|---|
| $d^5f$ | $+\ 486$ | | | $+\ 486$ |
| $d^3e^3$ | $+\ 36$ | $+\ 54$ | $-\ 27$ | $+\ 63$ |
| $c^2d^4e$ | $-1296$ | $-486$ | $+324$ | $-1458$ |
| $cd^3e^2f$ | $-\ 324$ | $-324$ | $+216$ | $-\ 432$ |
| $cde^5$ | | | $+\ 1$ | $+\ 1$ |
| $c^3d^2e^3$ | $+\ 144$ | $+324$ | $-216$ | $+\ 252$ |
| $c^2de^4f$ | $+\ 6$ | $+\ 36$ | $-\ 24$ | $+\ 18$ |
| $c^4e^5$ | | $-\ 6$ | $+\ 4$ | $-\ 2$ |

where the last column is, in fact, what $V_1$ becomes on writing therein $a=1$, $b=0$. The verification would not of course apply to terms which contain $b$; thus (13.3) a derived syzygy is $jr=bt+mo$; and the foregoing values give, as they should do, $jr=mo$: we might for the verification of most of the terms in $b$ use values $a, b, c, d, e, f^2 =1, b, 0, d, e, -d$: the only failure would be for terms containing $bc$.

## Table No 97 (Covariants of A, in the *af*- or standard forms: W is not given).

The several covariants are—

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $A=($ | $1$ | $0$ | $c+10$ | $f+10$ | $a^2b+\ 5$<br>$c^2\ -15$ | $a^2e+1$<br>$cf\ -2$ | $\between x, y)^5$ |
| | 0.0 | 1.3 | 2.6 | 3.9 | 4.12 | 5.15 | |

| | | | | |
|---|---|---|---|---|
| $B=a^2($ | $b\ +1$ | $e\ +1$ | $ad\ -3$<br>$bc\ -1$ | $\between x, y)^2$ |
| | 2.2 | 3.5 | 4.8 | |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $C=($ | $c+1$ | $f+1$ | $a^2b+\ 3$<br>$c^2\ -15$ | $a^2e+\ 1$<br>$cf\ -10$ | $a^3d\ +\ 6$<br>$a^2bc-\ 3$<br>$c^3\ \ +15$ | $a^2bf-3$<br>,, $ce+3$<br>$a^0cf+3$ | $a^4b^2\ -1$<br>$a^3cd\ +2$<br>$a^2bc^2+4$<br>,, $ef\ +1$<br>$a^0c^4\ -1$ | $\between x, y)^6$ |
| | 2.6 | 3.9 | 4.12 | 5.15 | 6.18 | 7.21 | 8.24 | |

| | | | | | |
|---|---|---|---|---|---|
| $D=a^2($ | $ad\ +1$ | $bf\ -\ 1$<br>$ce\ +\ 1$ | $a^2b^2\ -\ 1$<br>$acd\ +\ 3$<br>$a^0bc^2+\ 4$<br>,, $cf\ +\ 1$ | $adf\ +\ 1$<br>$a^0bcf+\ 1$<br>,, $c^2e-\ 1$ | $\between x, y)^3$ |
| | 4.8 | 5.11 | 6.14 | 7.17 | |

Table No. 97 (continued).

| $E=a^2($ | | | | | | | |
|---|---|---|---|---|---|---|---|
| | $e+1$ | $ad-6$<br>$a^0bc-10$ | $bf-12$<br>$ce+2$ | $a^2b^2-8$<br>$acd-36$<br>$a^0bc^2+12$<br>,, $ef-2$ | $a^2be-5$<br>$adf-24$<br>$a^0bcf-4$<br>,, $c^2e+2$ | $a^3bd-6$<br>$a^2b^2c-2$<br>,, $e^2-1$<br>$ac^2d+18$<br>$a^0bc^3+6$<br>,, $cef+2$ | $\between x, y)^5$ |
| | 3.5 | 4.8 | 5.11 | 6.14 | 7.17 | 8.20 | |

| $F=($ | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | $f+1$ | $a^2b+2$<br>$a^0c^2-18$ | $a^2e+1$<br>$a^0cf-36$ | $a^3d+34$<br>$a^2bc-42$<br>$a^0c^3+168$ | $a^2bf-40$<br>,, $ce+5$<br>$a^0c^2f-126$ | $a^4b^2-16$<br>$a^3cd+6$<br>$a^2bc^2+134$<br>,, $ef-5$<br>$a^0c^4-252$ | $a^4be-7$<br>$a^3df+8$<br>$a^2bcf+8$<br>,, $c^2e+55$<br>$a^0c^3f-84$ | $a^5bd+6$<br>$a^4b^2c-22$<br>,, $e^2-1$<br>$a^3c^2d+54$<br>$a^2bc^2+66$<br>,, $cef+38$<br>$a^0c^5+72$ | $a^5de-12$<br>$a^4bce+11$<br>,, $b^2f-9$<br>$a^3cdf+24$<br>$a^2bc^2f+32$<br>,, $c^3e-45$<br>$a^0c^4f+9$ | $a^6b^3-2$<br>$a^5bcd+6$<br>$a^4b^2c^2+12$<br>,, $bef+3$<br>,, $ce^2-1$<br>$a^3c^3d-14$<br>$a^2bc^4-16$<br>,, $c^2ef-5$<br>$a^0c^6-2$ | $\between x, y)^9$ |
| | 3.9 | 4.12 | 5.15 | 6.18 | 7.21 | 8.24 | 9.27 | 10.30 | 11.33 | 12.36 | |

| $G=a^4$ | |
|---|---|
| | $abd+12$<br>$a^0b^2c+4$<br>,, $e^2+1$<br>$=a^2g$ |
| | 6.10 |

| $H=a^2($ | | | | | | |
|---|---|---|---|---|---|---|
| | $acd+6$<br>$a^0bc^2+4$<br>,, $ef+1$<br>$=a^2h$ | $a^2be+2$<br>$adf+12$<br>$a^0bcf-8$<br>,, $c^2e-8$ | $a^2b^2c+4$<br>,, $e^2+1$<br>$ac^2d-36$<br>$a^0bc^3-24$<br>,, $cef-6$ | $a^3de+2$<br>$a^2b^2f+4$<br>,, $bce-6$<br>$acdf-12$<br>$a^0bc^2f-8$<br>,, $c^3e+8$ | $a^4b^3+2$<br>,, $d^2+6$<br>$a^3bcd-2$<br>$a^2b^2c^2-8$<br>,, $bef-3$<br>,, $ce^2+1$<br>$ac^3d+6$<br>$a^0bc^4+4$<br>,, $c^2ef+1$ | $\between x, y)^4$ |
| | 6.14 | 7.17 | 8.20 | 9.23 | 10.26 | |

| $I=a^2($ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | $bf-1$<br>$ce+1$<br>$=ai$ | $a^2b^2-2$<br>$acd-6$<br>$a^0bc^2+8$<br>,, $ef-2$ | $adf-15$<br>$a^0bcf+5$<br>,, $c^2e-5$ | $a^3bd-20$<br>$ac^2d+60$ | $a^3de-5$<br>$a^2b^2f+5$<br>,, $bce-5$<br>$acdf+30$<br>$a^0bc^2f+5$<br>,, $c^3e-5$ | $a^4b^3+2$<br>,, $d^2-12$<br>$a^3bcd-2$<br>$a^2b^2c^2-6$<br>,, $ce^2-2$<br>$ac^3d-30$<br>$a^0bc^4-8$<br>,, $c^2ef-2$ | $a^4b^2e+1$<br>$a^3bdf+3$<br>,, $cde-5$<br>$a^2b^2cf+1$<br>,, $bc^2e-5$<br>,, $e^2f-1$<br>$ac^2df-3$<br>$a^0bc^3f-1$<br>,, $c^4e+1$ | $\between x, y)^6$ |
| | 5.11 | 6.14 | 7.17 | 8.20 | 9.23 | 10.26 | 11.29 | |

## Table No. 97 (continued).

$J = a^4($

| | |
|---|---|
| $a^2b^3$ + 1 | $a^2b^2e$ − 1 |
| ,, $d^2$ + 9 | $abdf$ − 6 |
| $a^0b^2c^2$ − 4 | ,, $cde$ + 6 |
| ,, $bef$ − 2 | $a^0b^2cf$ − 4 |
| ,, $ce^2$ + 1 | ,, $bc^2e$ + 8 |
| | ,, $e^2f$ + 1 |
| $=a^3j$ | |
| 8.16 | 9.19 |

$)(x, y)^1$

$K = a^4($

| | | | |
|---|---|---|---|
| $ade$ + 3 | $a^2b^3$ + 4 | $a^2b^2e$ + 1 | $a^3b^2d$ + 6 |
| $a^0b^2f$ + 2 | ,, $d^2$ − 18 | $abdf$ + 6 | $a^2b^3c$ + 2 |
| ,, $bce$ − 2 | $abcd$ − 18 | $acde$ − 15 | ,, $cd^2$ − 18 |
| | $a^0b^2c^2$ − 16 | $a^0b^2cf$ − 2 | $abc^2d$ − 30 |
| | ,, $bef$ − 5 | ,, $bc^2e$ − 2 | $adef$ − 9 |
| | ,, $ce^2$ + 1 | ,, $e^2f$ − 1 | $a^0b^2c^3$ − 8 |
| | | | ,, $bcef$ − 5 |
| $=a^2k$ | | | ,, $c^2e^2$ + 3 |
| 7.13 | 8.16 | 9.19 | 10.22 |

$)(x, y)^3$

$L = a^2($

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $adf$ − 3 | $a^3bd$ − 3 | $a^3de$ − 12 | $a^4b^3$ − 6 | $a^4b^2e$ − 1 | $a^5b^2d$ + 15 | $a^5bde$ − 7 | $a^6b^4$ − 2 |
| $a^0bcf$ − 2 | $a^2b^2c$ − 7 | $a^2b^2f$ − 9 | ,, $d^2$ − 39 | $a^3bdf$ + 39 | $a^4b^3c$ − 9 | $a^4b^3f$ − 7 | ,, $bd^2$ + 3 |
| ,, $c^2e$ + 2 | $ac^2d$ + 42 | ,, $bce$ + 9 | $a^3bcd$ + 40 | ,, $cde$ − 14 | ,, $cd^2$ + 18 | ,, $b^2ce$ + 14 | $a^5b^2cd$ + 10 |
| | $a^0bc^3$ + 28 | $acdf$ + 63 | $a^2b^2c^2$ + 59 | $a^2b^2cf$ + 16 | $a^3bc^2d$ − 33 | ,, $d^2f$ + 12 | ,, $de^2$ + 2 |
| | ,, $cef$ + 7 | $a^0bc^2f$ + 42 | ,, $bef$ + 7 | ,, $bc^2e$ − 12 | ,, $def$ − 3 | $a^3bcdf$ + 23 | $a^4b^3c^2$ + 13 |
| | | ,, $c^3e$ − 42 | ,, $ce^2$ − 1 | ,, $e^2f$ + 1 | $a^2b^2c^3$ + 15 | ,, $c^2de$ − 26 | ,, $b^2ef$ + 4 |
| | | | $ac^3d$ − 210 | $ac^2df$ − 105 | ,, $bcef$ + 21 | $a^2b^2c^2f$ + 25 | ,, $bce^2$ − 2 |
| | | | $a^0bc^4$ − 140 | $a^0bc^3f$ − 70 | ,, $c^2e^2$ − 12 | ,, $bc^3e$ − 53 | ,, $c^2d^2$ − 15 |
| | | | ,, $c^2ef$ − 35 | ,, $c^4e$ + 70 | $ac^4d$ + 126 | ,, $ce^2f$ − 7 | $a^3bc^3d$ − 28 |
| | | | | | $a^0bc^5$ + 84 | $ac^3df$ + 21 | ,, $cdef$ − 7 |
| | | | | | ,, $c^3ef$ + 21 | $a^0bc^4f$ + 14 | $a^2b^2c^4$ − 19 |
| | | | | | | ,, $c^5e$ − 14 | ,, $bc^2ef$ − 10 |
| | | | | | | | ,, $c^3e^2$ + 5 |
| | | | | | | | $ac^5d$ − 6 |
| $=a^2l$ | | | | | | | $a^0bc^6$ − 4 |
| | | | | | | | ,, $c^4ef$ − 1 |
| 7.17 | 8.20 | 9.23 | 10.26 | 11.29 | 12.32 | 13.35 | 14.38 |

$)(x, y)^7$

$M = a^4($

| | | |
|---|---|---|
| $a^3b^2d$ − 2 | $a^3bde$ − 1 | $a^4b^4$ − 1 |
| $a^2b^3c$ − 1 | $a^2b^3f$ − 1 | ,, $bd^2$ + 3 |
| ,, $cd^2$ + 9 | ,, $b^2ce$ + 2 | $a^3b^2cd$ + 6 |
| $abc^2d$ + 12 | ,, $d^2f$ + 9 | ,, $de^2$ + 1 |
| ,, $def$ + 3 | $abcdf$ + 12 | $a^2b^3c^2$ + 5 |
| $a^0b^2c^3$ + 4 | ,, $c^2de$ − 12 | ,, $b^2ef$ + 2 |
| ,, $bcef$ + 2 | $a^0b^2c^2f$ + 4 | ,, $bce^2$ − 1 |
| ,, $c^2e^2$ − 1 | ,, $bc^3e$ − 8 | ,, $c^2d^2$ − 9 |
| | ,, $ce^2f$ − 1 | $abc^3d$ − 12 |
| | | ,, $cdef$ − 3 |
| | | $a^0b^2c^4$ − 4 |
| | | ,, $bc^2ef$ − 2 |
| $=a^4m$ | | ,, $c^3e^2$ + 1 |
| 10.22 | 11.25 | 12.28 |

$)(x, y)^2$

Table No. 97 (continued).

$\mathrm{N}=a^4($

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $a^2b^2e$ | $+1$ | $a^2b^3c$ | $+4$ | $a^3bde$ | $+12$ | $a^4b^4$ | $+4$ | $a^4b^3e$ | $+2$ |
| $a\,bdf$ | $+6$ | ,, $cd^2$ | $+36$ | $a^2b^3f$ | $+6$ | ,, $bd^2$ | $+12$ | ,, $d^2e$ | $+9$ |
| ,, $cde$ | $-6$ | $a^0b^2c^3$ | $-16$ | ,, $b^2ce$ | $-6$ | $a^3b^2cd$ | $+8$ | $a^3b^2df$ | $+6$ |
| $a^0b^2cf$ | $+4$ | ,, $bcef$ | $-8$ | ,, $d^2f$ | $+54$ | ,, $de^2$ | $+4$ | ,, $bcde$ | $-10$ |
| ,, $bc^2e$ | $-8$ | ,, $c^2e^2$ | $+4$ | $abcdf$ | $+36$ | $a^2b^3c^2$ | $-12$ | $a^2b^3cf$ | $-2$ |
| ,, $e^2f$ | $-1$ | | | ,, $c^2de$ | $-36$ | ,, $b^2ef$ | $-4$ | ,, $b^2c^2e$ | $-11$ |
| | | | | | | ,, $c^2d^2$ | $-108$ | ,, $be^2f$ | $-3$ |
| | | | | | | $abc^3d$ | $-96$ | ,, $cd^2f$ | $-18$ |
| | | | | | | ,, $cdef$ | $-24$ | ,, $ce^3$ | $+1$ |
| | | | | | | $a^0b^2c^4$ | $-16$ | $abc^2df$ | $-18$ |
| | | | | | | ,, $bc^2ef$ | $-8$ | $c^3de$ | $+18$ |
| | | | | | | ,, $c^3e^2$ | $+4$ | $a^0b^2c^3f$ | $-4$ |
| | | | | | | | | $bc^4e$ | $+8$ |
| $=a^3n$ | | | | | | | | $ce^2f$ | $+1$ |
| 9.19 | | 10.22 | | 11.25 | | 12.28 | | 13.31 | |

$)(x, y)^4$

$\mathrm{O}=a^6($

| | | | |
|---|---|---|---|
| $a^2b^3e$ | $+3$ | $a^3b^3d$ | $-6$ |
| ,, $d^2e$ | $+9$ | ,, $d^3$ | $-54$ |
| $ab^2df$ | $+12$ | $a^2b^4c$ | $-2$ |
| ,, $bcde$ | $-12$ | ,, $b^2e^2$ | $+1$ |
| $a^0b^3cf$ | $+8$ | ,, $bcd^2$ | $-18$ |
| ,, $b^2c^2e$ | $-20$ | $ab^2c^2d$ | $+24$ |
| ,, $be^2f$ | $-4$ | ,, $bdef$ | $+18$ |
| ,, $ce^3$ | $+1$ | ,, $cde^2$ | $-12$ |
| | | $a^0b^3c^3$ | $+8$ |
| | | ,, $b^2cef$ | $-8$ |
| | | ,, $bc^2e^2$ | $-10$ |
| $=a^4o$ | | ,, $e^3f$ | $-1$ |
| 11.21 | | 12.24 | |

$)(x, y)^1$

$\mathrm{P}=a^4($

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $a^3bde$ | $-2$ | $a^4b^4$ | $-2$ | $a^4b^3e$ | $-5$ | $a^5b^3d$ | $-8$ | $a^5b^2de$ | $+6$ | $a^6b^5$ | $+2$ |
| $a^2b^3f$ | $-1$ | ,, $bd^2$ | $+12$ | ,, $d^2e$ | $-9$ | ,, $d^3$ | $-72$ | $a^4b^4f$ | $+5$ | ,, $b^2d^2$ | $+12$ |
| ,, $d^2f$ | $-9$ | $a^3b^2cd$ | $-10$ | $a^3b^2df$ | $-24$ | $a^4b^4c$ | $+4$ | ,, $b^3ce$ | $-13$ | $a^5b^3cd$ | $-2$ |
| $a\,bcdf$ | $-12$ | ,, $de^2$ | $-5$ | ,, $bcde$ | $+44$ | ,, $b^2e^2$ | $-1$ | ,, $bd^2f$ | $+21$ | ,, $bde^2$ | $-3$ |
| ,, $c^2de$ | $+12$ | $a^2b^3c^2$ | $-2$ | $a^2b^3cf$ | $-6$ | ,, $bcd^2$ | $-24$ | ,, $cd^2e$ | $-21$ | ,, $cd^3$ | $-36$ |
| $a^0b^2c^2f$ | $-4$ | ,, $b^2e^2f$ | $-1$ | ,, $b^2c^2e$ | $+36$ | $a^3b^2cd$ | $+52$ | $a^3b^2cdf$ | $-4$ | $a^4b^4c^2$ | $-12$ |
| ,, $bc^3e$ | $+8$ | ,, $bce^2$ | $+3$ | ,, $be^2f$ | $+6$ | ,, $bdef$ | $+10$ | ,, $bc^2de$ | $+10$ | ,, $b^3ef$ | $-5$ |
| ,, $ce^2f$ | $+1$ | ,, $c^2d^2$ | $+90$ | ,, $cd^2f$ | $+90$ | ,, $cde^2$ | $+8$ | ,, $de^2f$ | $+4$ | ,, $b^2ce^2$ | $+4$ |
| | | $abc^3d$ | $+120$ | ,, $ce^3$ | $-1$ | $a^2b^3c^3$ | $+4$ | $a^2b^3c^2f$ | $-17$ | ,, $bc^2d^2$ | $-66$ |
| | | ,, $cdef$ | $+30$ | $abc^2df$ | $+120$ | ,, $b^2cef$ | $-2$ | ,, $b^2c^3e$ | $+44$ | ,, $d^2ef$ | $-18$ |
| | | $a^0b^2c^4$ | $+40$ | ,, $c^3de$ | $-120$ | ,, $bc^2e^2$ | $+2$ | ,, $bce^2f$ | $+13$ | $a^3b^2c^3d$ | $-10$ |
| | | ,, $bc^2ef$ | $+20$ | $a^0b^2c^3f$ | $+40$ | ,, $c^3d^2$ | $-180$ | ,, $c^2d^2f$ | $-45$ | ,, $bcdef$ | $-4$ |
| | | ,, $c^3e^2$ | $-10$ | ,, $bc^4e$ | $-80$ | ,, $e^3f$ | $+1$ | ,, $c^2e^3$ | $-5$ | ,, $c^2de^2$ | $+1$ |
| | | | | ,, $c^2e^2f$ | $-10$ | $a\,bc^4d$ | $-240$ | $abc^3df$ | $-60$ | $a^2b^3c^4$ | $+14$ |
| | | | | | | ,, $c^2def$ | $-60$ | ,, $c^4de$ | $+60$ | ,, $b^2c^2ef$ | $+11$ |
| | | | | | | $a^0b^2c^5$ | $-80$ | $a^0b^2c^4f$ | $-20$ | ,, $bc^3e^2$ | $-11$ |
| | | | | | | ,, $bc^3ef$ | $-40$ | ,, $bc^5e$ | $+40$ | ,, $c^4d^2$ | $+18$ |
| | | | | | | ,, $c^4e^2$ | $+20$ | ,, $c^3e^2f$ | $-5$ | ,, $ce^3f$ | $-1$ |
| | | | | | | | | | | $abc^5d$ | $+26$ |
| | | | | | | | | | | ,, $c^3def$ | $+6$ |
| | | | | | | | | | | $a^0b^2c^6$ | $+8$ |
| | | | | | | | | | | ,, $bc^4ef$ | $+4$ |
| $=a^4p$ | | | | | | | | | | ,, $c^5e^2$ | $-2$ |
| 11.25 | | 12.28 | | 13.31 | | 14.34 | | 15.37 | | 16.40 | |

$)(x, y)^5$

## Table No. 97 (continued).

$Q = a^6($

| | |
|---|---|
| $a^4b^5$ | $-\ 2$ |
| ,, $b^2d^2$ | $+\ 18$ |
| $a^3b^3cd$ | $+\ 22$ |
| ,, $bde^2$ | $+\ 3$ |
| ,, $cd^3$ | $-\ 54$ |
| $a^2b^4c^2$ | $+\ 12$ |
| ,, $b^3ef$ | $+\ 5$ |
| ,, $b^2ce^2$ | $-\ 4$ |
| ,, $bc^2d^2$ | $-108$ |
| ,, $d^2ef$ | $-\ 27$ |
| $a\ b^2c^3d$ | $-\ 72$ |
| ,, $bcdef$ | $-\ 36$ |
| ,, $c^2de^2$ | $+\ 18$ |
| $a^0b^3c^4$ | $-\ 16$ |
| ,, $b^2c^2ef$ | $-\ 12$ |
| ,, $bc^3e^2$ | $+\ 12$ |
| ,, $ce^3f$ | $+\ 1$ |
| $=a^6q$ | |
| 14.30 | |

$R = a^6($

| | | | | | |
|---|---|---|---|---|---|
| $a^3bde$ | $-\ 1$ | $a^4b^5$ | $+\ 2$ | $a^4b^4e$ | $+\ 1$ |
| $a^2b^4f$ | $+\ 1$ | ,, $b^2d^2$ | $+\ 6$ | $a^3b^3df$ | $+\ 3$ |
| ,, $b^3ce$ | $-\ 3$ | $a^3b^3cd$ | $-\ 2$ | ,, $b^2cde$ | $-\ 11$ |
| ,, $bd^2f$ | $-\ 9$ | ,, $bde^2$ | $-\ 2$ | ,, $d^3f$ | $-\ 27$ |
| ,, $cd^2e$ | $+\ 9$ | ,, $cd^3$ | $-\ 54$ | ,, $de^3$ | $-\ 1$ |
| $a\ b^2cdf$ | $-\ 12$ | $a^2b^4c^2$ | $-\ 8$ | $a^2b^4cf$ | $+\ 1$ |
| ,, $bc^2de$ | $+\ 24$ | ,, $b^3ef$ | $-\ 4$ | ,, $b^3c^2e$ | $-\ 7$ |
| ,, $de^2f$ | $+\ 3$ | ,, $b^2ce^2$ | $+\ 2$ | ,, $b^2e^2f$ | $-\ 2$ |
| $a^0b^3c^2f$ | $-\ 4$ | ,, $bc^2d^2$ | $-\ 72$ | ,, $bcd^2f$ | $-\ 45$ |
| ,, $b^2c^3e$ | $+\ 12$ | ,, $d^2ef$ | $-\ 18$ | ,, $bce^3$ | $+\ 1$ |
| ,, $bce^2f$ | $+\ 3$ | $ab^3c^3d$ | $-\ 24$ | ,, $c^2d^2e$ | $+\ 45$ |
| ,, $c^2e^3$ | $-\ 1$ | ,, $bcdef$ | $-\ 12$ | $a\ b^2c^2df$ | $-\ 24$ |
| | | ,, $c^2de^2$ | $+\ 6$ | ,, $bc^3de$ | $+\ 48$ |
| | | | | $a^0b^3cf$ | $-\ 4$ |
| | | | | ,, $b^2c^4e$ | $+\ 12$ |
| | | | | ,, $bc^2e^2f$ | $+\ 3$ |
| | | | | ,, $c^3e^3$ | $-\ 1$ |
| $=a^5r$ | | | | | |
| 13.27 | | 14.30 | | 15.33 | |

$)\!\!(x, y)^2$

$S = a^6($

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $a^4bd^2e$ | $+\ 9$ | $a^5b^4d$ | $+\ 15$ | $a^5b^3de$ | $-\ 6$ | $a^6b^6$ | $-\ 2$ |
| $a^3b^3df$ | $+\ 7$ | ,, $bd^3$ | $-\ 27$ | ,, $d^3e$ | $-\ 27$ | ,, $b^3d^2$ | $+\ 9$ |
| ,, $b^2cde$ | $-\ 12$ | $a^4b^5c$ | $+\ 3$ | $a^4b^5f$ | $-\ 3$ | ,, $d^4$ | $-\ 27$ |
| ,, $d^3f$ | $-\ 27$ | ,, $b^2cd^2$ | $-\ 99$ | ,, $b^4ce$ | $+\ 9$ | $a^5b^4cd$ | $+\ 18$ |
| $a^2b^4cf$ | $+\ 2$ | ,, $d^2e^2$ | $-\ 18$ | ,, $b^2d^2f$ | $-\ 9$ | ,, $b^2de^2$ | $+\ 6$ |
| ,, $b^3c^2e$ | $-\ 6$ | $a^3b^3c^2d$ | $-\ 114$ | ,, $bcd^2e$ | $-\ 18$ | ,, $bcd^3$ | $-\ 54$ |
| ,, $bcd^2f$ | $-\ 54$ | ,, $b^2def$ | $-\ 33$ | $a^3b^3cdf$ | $-\ 9$ | $a^4b^5c^2$ | $+\ 15$ |
| ,, $c^2d^2e$ | $+\ 54$ | ,, $bcde^2$ | $+\ 12$ | ,, $b^2c^2de$ | $+\ 24$ | ,, $b^4ef$ | $+\ 6$ |
| $a\ b^2c^2df$ | $-\ 36$ | ,, $c^2d^3$ | $+\ 162$ | ,, $bde^2f$ | $+\ 3$ | ,, $b^3c^2d^2$ | $-\ 36$ |
| ,, $bc^3de$ | $+\ 72$ | $a^2b^4c^2$ | $-\ 24$ | ,, $cd^3f$ | $+\ 81$ | ,, $b^3ce^2$ | $-\ 6$ |
| ,, $cde^2f$ | $+\ 9$ | ,, $b^3cef$ | $-\ 9$ | ,, $cde^3$ | $-\ 3$ | ,, $b^2c^2d^2$ | $-\ 27$ |
| $a^0b^3c^3f$ | $-\ 8$ | ,, $b^2c^2e^2$ | $+\ 9$ | $a^2b^4c^2f$ | $+\ 6$ | ,, $bd^2ef$ | $-\ 9$ |
| ,, $b^2c^4e$ | $+\ 24$ | ,, $bc^3d^2$ | $+\ 324$ | ,, $b^3c^3e$ | $-\ 18$ | ,, $cd^2e^2$ | $-\ 9$ |
| ,, $bc^2e^2f$ | $+\ 6$ | ,, $cd^2ef$ | $+\ 69$ | ,, $b^2ce^2f$ | $-\ 9$ | $a^3b^3c^3d$ | $-\ 54$ |
| ,, $c^3e^3$ | $-\ 2$ | $a\ b^2c^4d$ | $+\ 216$ | ,, $bc^2d^2f$ | $+\ 162$ | ,, $b^2cdef$ | $-\ 27$ |
| | | ,, $bc^2def$ | $+\ 120$ | ,, $bc^2e^3$ | $+\ 3$ | ,, $bc^2de^2$ | $+\ 3$ |
| | | ,, $c^3de^2$ | $-\ 54$ | ,, $c^3d^2e$ | $-\ 162$ | ,, $c^3d^3$ | $-\ 54$ |
| | | $a^0b^3c^5$ | $+\ 48$ | $a\ b^2c^3df$ | $+\ 108$ | ,, $de^3f$ | $-\ 2$ |
| | | ,, $b^2c^3ef$ | $+\ 36$ | ,, $bc^4de$ | $-\ 216$ | $a^2b^4c^4$ | $-\ 24$ |
| | | ,, $bc^4e^2$ | $-\ 36$ | ,, $c^2de^2f$ | $-\ 27$ | ,, $b^3ce^2f$ | $-\ 21$ |
| | | ,, $c^2e^3f$ | $-\ 3$ | $a^0b^3c^4f$ | $+\ 24$ | ,, $b^2c^3e^2$ | $+\ 21$ |
| | | | | ,, $b^2c^5e$ | $-\ 72$ | ,, $bc^4d^2$ | $-\ 108$ |
| | | | | ,, $bc^3e^2f$ | $-\ 18$ | ,, $bce^3f$ | $+\ 2$ |
| | | | | ,, $c^4e^3$ | $+\ 6$ | ,, $c^2d^2ef$ | $-\ 27$ |
| | | | | | | $a\ b^2c^5d$ | $-\ 72$ |
| | | | | | | ,, $bc^3def$ | $-\ 36$ |
| | | | | | | ,, $c^4de^2$ | $+\ 18$ |
| | | | | | | $a^0b^3c^6$ | $-\ 16$ |
| | | | | | | ,, $b^2c^4ef$ | $-\ 12$ |
| | | | | | | ,, $bc^5e^2$ | $+\ 12$ |
| $=a^6s$ | | | | | | ,, $c^3e^3f$ | $+\ 1$ |
| 15.33 | | 16.36 | | 17.39 | | 18.42 | |

$)\!\!(x, y)^3$

Table No. 97 (continued).

$T=($

| | | | | | |
|---|---|---|---|---|---|
| $a^5b^4de$ | $+$ | 7 | $a^6b^7$ | $+$ | 2 |
| ,, $bd^3e$ | $+$ | 27 | ,, $b^4d^2$ | $+$ | 6 |
| $a^4b^6f$ | $+$ | 1 | $a^5b^5cd$ | $-$ | 10 |
| ,, $b^5ce$ | $-$ | 2 | ,, $b^3de^2$ | $-$ | 8 |
| ,, $b^3d^2f$ | $+$ | 24 | ,, $b^2cd^2$ | $-$ | 54 |
| ,, $b^2cd^2e$ | $-$ | 54 | ,, $d^3e^2$ | $-$ | 27 |
| ,, $d^4f$ | $-$ | 81 | $a^4b^6c^2$ | $-$ | 14 |
| ,, $d^2e^3$ | $-$ | 6 | ,, $b^5ef$ | $-$ | 7 |
| $a^3b^4cdf$ | $+$ | 16 | ,, $b^4ce^2$ | $+$ | 9 |
| ,, $b^3c^2de$ | $-$ | 76 | ,, $b^3c^2d^2$ | $-$ | 84 |
| ,, $b^2de^2f$ | $-$ | 12 | ,, $b^2d^2ef$ | $-$ | 27 |
| ,, $bcd^3f$ | $-$ | 216 | ,, $bcd^2e^2$ | $+$ | 9 |
| ,, $bcde^3$ | $+$ | 5 | ,, $c^2d^4$ | $+$ | 162 |
| ,, $c^2d^3e$ | $+$ | 216 | $a^3b^4c^3d$ | $-$ | 8 |
| $a^2b^4c^3e$ | $-$ | 8 | ,, $b^3cdef$ | $+$ | 4 |
| ,, $b^2c^2d^2f$ | $-$ | 216 | ,, $b^2c^2de^2$ | $+$ | 18 |
| ,, $b^2c^2e^3$ | $+$ | 2 | ,, $bc^3d^3$ | $+$ | 432 |
| ,, $bc^3d^2e$ | $+$ | 432 | ,, $bde^3f$ | $+$ | 3 |
| ,, $cd^2e^2f$ | $+$ | 54 | ,, $cd^3ef$ | $+$ | 108 |
| $a\ b^3c^3df$ | $-$ | 96 | ,, $cde^4$ | $-$ | 1 |
| ,, $b^2c^4de$ | $+$ | 288 | $a^2b^5c^4$ | $+$ | 16 |
| ,, $bc^2de^2f$ | $+$ | 72 | ,, $b^4c^2ef$ | $+$ | 20 |
| ,, $c^3de^3$ | $-$ | 24 | ,, $b^3c^3e^2$ | $-$ | 24 |
| $a^0b^4c^4f$ | $-$ | 16 | ,, $b^2c^4d^2$ | $+$ | 432 |
| ,, $b^3c^5e$ | $+$ | 64 | ,, $b^2ce^3f$ | $-$ | 5 |
| ,, $b^2c^3e^2f$ | $+$ | 24 | ,, $bc^2d^2ef$ | $+$ | 216 |
| ,, $bc^4e^3$ | $-$ | 16 | ,, $bc^2e^4$ | $+$ | 1 |
| ,, $c^2e^4f$ | $-$ | 1 | ,, $c^3d^2e^2$ | $-$ | 108 |
| | | | $a\ b^3c^5d$ | $+$ | 192 |
| | | | ,, $b^2c^3def$ | $+$ | 144 |
| | | | ,, $bc^4de^2$ | $-$ | 144 |
| | | | ,, $c^2de^3f$ | $-$ | 12 |
| | | | $a^0b^4c^6$ | $+$ | 32 |
| | | | ,, $b^3c^4ef$ | $+$ | 32 |
| | | | ,, $b^2c^5e^2$ | $-$ | 48 |
| | | | ,, $bc^3e^3f$ | $-$ | 8 |
| $=a^8t$ | | | ,, $c^4e^4$ | $+$ | 2 |
| 19.41 | | | 20.44 | | |

$\big)(x, y)^1$

$U=$

| | | |
|---|---|---|
| $a^5b^6d$ | $-$ | 3 |
| ,, $b^3d^3$ | $+$ | 14 |
| ,, $d^5$ | $-$ | 27 |
| $a^4b^7c$ | $-$ | 1 |
| ,, $b^4cd^2$ | $+$ | 34 |
| ,, $b^2d^2e^2$ | $+$ | 11 |
| ,, $bcd^4$ | $-$ | 81 |
| $a^3b^5c^2d$ | $+$ | 32 |
| ,, $b^4def$ | $+$ | 10 |
| ,, $b^3cde^2$ | $-$ | 6 |
| ,, $b^2c^2d^3$ | $-$ | 144 |
| ,, $bd^3ef$ | $-$ | 18 |
| ,, $cd^3e^2$ | $-$ | 18 |
| $a^2b^6c^3$ | $+$ | 8 |
| ,, $b^5cef$ | $+$ | 4 |
| ,, $b^4c^2e^2$ | $-$ | 6 |
| ,, $b^3c^3d^2$ | $-$ | 152 |
| ,, $b^2cd^2ef$ | $-$ | 60 |
| ,, $bc^2d^2e^2$ | $+$ | 6 |
| ,, $d^2e^3f$ | $-$ | 4 |
| $a\ b^4c^4d$ | $-$ | 80 |
| ,, $b^3c^2def$ | $-$ | 56 |
| ,, $b^2c^3de^2$ | $+$ | 48 |
| ,, $bcde^3f$ | $+$ | 2 |
| ,, $c^2de^4$ | $+$ | 1 |
| $a^0b^5c^5$ | $-$ | 16 |
| ,, $b^4c^3ef$ | $+$ | 16 |
| ,, $b^3c^4e^2$ | $+$ | 24 |
| ,, $b^2c^2e^3f$ | $+$ | 4 |
| ,, $bc^3e^4$ | $+$ | 1 |
| $=a^9u$ | | |
| 21·45 | | |

## Table No. 97 (concluded).

$\mathrm{V}=a^{10}($

| | | | | | |
|---|---|---|---|---|---|
| $a^6b^8$ | $-$ | 4 | $a^6b^7e$ | $-$ | 2 |
| ,, $b^5d^2$ | $-$ | 12 | ,, $b^4d^2e$ | $-$ | 48 |
| $a^5b^6cd$ | $+$ | 20 | ,, $bd^4e$ | $-$ | 162 |
| ,, $b^4de^2$ | $+$ | 23 | $a^5b^6df$ | $-$ | 6 |
| ,, $b^3cd^3$ | $+$ | 108 | ,, $b^5cde$ | $+$ | 8 |
| ,, $bd^3e^2$ | $+$ | 81 | ,, $b^3d^3f$ | $-$ | 144 |
| $a^4b^7c^2$ | $+$ | 28 | ,, $b^3de^3$ | $+$ | 8 |
| ,, $b^6ef$ | $+$ | 15 | ,, $b^2cd^3e$ | $+$ | 324 |
| ,, $b^5ce^2$ | $-$ | 20 | ,, $d^5f$ | $+$ | 486 |
| ,, $b^4c^2d^2$ | $+$ | 168 | ,, $d^3e^3$ | $+$ | 63 |
| ,, $b^3d^2ef$ | $+$ | 78 | $a^4b^7cf$ | $-$ | 2 |
| ,, $b^2cd^2e^2$ | $-$ | 72 | ,, $b^6c^2e$ | $+$ | 18 |
| ,, $bc^2d^4$ | $-$ | 324 | ,, $b^5e^2f$ | $+$ | 7 |
| ,, $d^4ef$ | $-$ | 81 | ,, $b^4cd^2f$ | $-$ | 144 |
| ,, $d^2e^4$ | $-$ | 6 | ,, $b^4ce^3$ | $-$ | 9 |
| $a^3b^5c^3d$ | $+$ | 16 | ,, $b^3c^2d^2e$ | $+$ | 648 |
| ,, $b^4cdef$ | $+$ | 8 | ,, $b^2d^2e^2f$ | $+$ | 99 |
| ,, $b^3c^2de^2$ | $-$ | 112 | ,, $bcd^4f$ | $+$ | 1458 |
| ,, $b^2c^3d^3$ | $-$ | 864 | ,, $bcd^2e^3$ | $-$ | 27 |
| ,, $b^2de^3f$ | $-$ | 18 | ,, $c^2d^4e$ | $-$ | 1458 |
| ,, $bcd^3ef$ | $-$ | 432 | $a^3b^5c^2df$ | $-$ | 32 |
| ,, $bcde^4$ | $+$ | 7 | ,, $b^4c^3de$ | $+$ | 208 |
| ,, $c^2d^3e^2$ | $+$ | 216 | ,, $b^3cde^2f$ | $+$ | 20 |
| $a^2b^6c^4$ | $-$ | 32 | ,, $b^2c^2d^3f$ | $+$ | 1728 |
| ,, $b^5c^2ef$ | $-$ | 40 | ,, $b^2c^2de^3$ | $-$ | 40 |
| ,, $b^4c^3e^2$ | $+$ | 40 | ,, $bc^3d^3e$ | $-$ | 3456 |
| ,, $b^3c^4d^2$ | $-$ | 864 | ,, $bde^4f$ | $-$ | 3 |
| ,, $b^3ce^3f$ | $+$ | 10 | ,, $cd^3e^2f$ | $-$ | 432 |
| ,, $b^2c^2d^2ef$ | $-$ | 648 | ,, $cde^5$ | $+$ | 1 |
| ,, $bc^3d^2e^2$ | $+$ | 648 | $a^2b^4c^2e^2f$ | $-$ | 20 |
| ,, $cd^2e^3f$ | $+$ | 54 | ,, $b^3c^3d^2f$ | $+$ | 1008 |
| $a\ b^4c^5d$ | $-$ | 384 | ,, $b^3c^3e^3$ | $+$ | 20 |
| ,, $b^3c^3def$ | $-$ | 384 | ,, $b^2c^4d^2e$ | $-$ | 3024 |
| ,, $b^2c^4de^2$ | $+$ | 576 | ,, $b^2ce^4f$ | $+$ | 5 |
| ,, $bc^2de^3f$ | $+$ | 96 | ,, $bc^2d^2e^2f$ | $-$ | 756 |
| ,, $c^3de^4$ | $-$ | 24 | ,, $bc^2e^5$ | $-$ | 1 |
| $a^0b^5c^6$ | $-$ | 64 | ,, $c^3d^2e^3$ | $+$ | 252 |
| ,, $b^4c^4ef$ | $-$ | 80 | $a\ b^4c^4df$ | $+$ | 288 |
| ,, $b^3c^5e^2$ | $+$ | 160 | ,, $b^3c^5de$ | $-$ | 1152 |
| ,, $b^2c^3e^3f$ | $+$ | 40 | ,, $b^2c^3de^2f$ | $-$ | 432 |
| ,, $bc^4e^4$ | $-$ | 20 | ,, $bc^4de^3$ | $+$ | 288 |
| ,, $c^2e^5f$ | $-$ | 1 | ,, $c^2de^4f$ | $+$ | 18 |
| | | | $a^0b^5cf$ | $+$ | 32 |
| | | | ,, $b^4c^6e$ | $-$ | 160 |
| | | | ,, $b^3c^4e^2f$ | $-$ | 80 |
| | | | ,, $b^2c^5e^3$ | $+$ | 80 |
| | | | ,, $bc^3e^4f$ | $+$ | 10 |
| $=a^9v$ | | | ,, $c^4e^5$ | $-$ | 2 |
| 22.46 | | | 23.49 | | |

$\between x, y)^2$

Table No. 98. Covariants of A, divided and (except as to a few coefficients) segregate.

A and B as given in Table 97 were divided and segregate.

C was divided but not segregate : the divided and segregate form is

$\mathrm{C}=($

| | | | | | | |
|---|---|---|---|---|---|---|
| $c+1$ | $f+3$ | $a^2b + 3$<br>$a^0c^2 - 15$ | $a^2e + 1$<br>$a^0cf - 10$ | $a^3d + 6$<br>$a^0bc - 3$<br>,, $c^3 + 15$ | $a^3i + 3$<br>$a^0c^2f + 3$ | $a^4b^2 - 1$<br>,, $h + 1$<br>$a^3cd - 4$<br>$a^0c^4 - 1$ |
| 2.6 | 3.9 | 4.12 | 5.15 | 6.18 | 7.21 | 8.24 |

$)(x, y)^6$

D divided and segregate is

$\mathrm{D}=a^3($

| | | | ÷3 |
|---|---|---|---|
| $d + 1$ | $i + 1$ | $ab^2 - 1$<br>,, $h + 1$<br>$a^0cd - 3$ | $al - 1$<br>$a^0ci - 1$ |
| 3.3 | 4.6 | 5.9 | 6.12 |

$)(x, y)^3,$

an integer non-segregate form of the fractional coefficient is

| |
|---|
| $ci - 1$<br>$df + 1$ |

E was divided but not segregate : the divided and segregate form is

$\mathrm{E}=($

| | | | | | |
|---|---|---|---|---|---|
| $e + 1$ | $ad - 6$<br>$a^0bc - 10$ | $ai + 12$<br>$a^0ce - 10$ | $a^2b^2 - 8$<br>,, $h - 2$<br>$acd - 24$<br>$a^0bc^2 + 20$ | $a^2be - 5$<br>,, $l + 8$<br>$aci - 12$<br>$a^0c^2e + 5$ | $a^4g - 1$<br>$a^3bd + 6$<br>$a^2b^2c + 2$<br>,, $ch + 2$<br>$ac^2d + 6$<br>$a^0bc^3 - 2$ |
| 3.5 | 4.8 | 5.11 | 6.14 | 7.17 | 8.20 |

$)(x, y)^5.$

F was divided but not segregate : the divided and segregate form is

$\mathrm{F}=($

| | | | | | | ÷3 | | | |
|---|---|---|---|---|---|---|---|---|---|
| $f+1$ | $a^2b + 2$<br>$a^0c^2 - 18$ | $a^2e + 1$<br>$a^0cf - 36$ | $a^3d + 34$<br>$a^2bc - 42$<br>$a^0c^3 + 168$ | $a^3i + 40$<br>$a^2ce - 35$<br>$a^0c^2f + 126$ | $a^4b^2 - 16$<br>,, $h - 5$<br>$a^3cd + 46$<br>$a^2bc^2 + 155$<br>$a^0c^4 + 252$ | $a^4be - 21$<br>$a^3l - 8$<br>,, $ci - 16$<br>$a^2c^2e + 189$<br>$a^0c^3f - 252$ | $a^6g - 1$<br>$a^5bd + 18$<br>$a^4b^2c - 18$<br>,, $ch + 38$<br>$a^3c^2d - 174$<br>$a^2bc^2 - 86$<br>$a^0c^5 + 72$ | $a^6k - 4$<br>$a^5bi + 1$<br>$a^4bce + 2$<br>,, $cl - 8$<br>$a^3c^2i - 16$<br>$a^2c^3e - 13$<br>$a^0c^4f + 9$ | $a^6b^3 - 2$<br>,, $bh + 3$<br>,, $cg - 1$<br>$a^4b^2c^2 + 4$<br>,, $c^2h - 5$<br>$a^3c^3d + 16$<br>$a^2bc^4 + 4$<br>$a^0c^6 - 2$ |
| 3.9 | 4.12 | 5.15 | 6.18 | 7.21 | 8.24 | 9.27 | 10.30 | 11.33 | 12.36 |

$)(x; y)^9,$

where for an integer non-segregate value of the fractional coefficient, see the original form of F.

Table No. 98 (continued).

G as an invariant was divided and segregate, $G = a^6 \underset{4.0}{g}$

H divided and segregate is

|  |  |  |  | ÷3 | ÷3 |  |
|---|---|---|---|---|---|---|
| $H = a^4($ | $h$ + 1 | $be$ +2<br>$l$ −4 | $a^2g$ + 1<br>$abd$ −12<br>$a^0ch$ − 6 | $a^2k$ + 2<br>$abi$ − 8<br>$a^0bce$ − 6<br>,, $cl$ +12 | $a^3j$ + 2<br>$a^2b^3$ + 4<br>,, $bh$ − 5<br>,, $cg$ + 1<br>$abcd$ +12<br>$a^0c^2h$ + 3 | $)(x, y)^4,$ |
|  | 4.4 | 5.7 | 6.10 | 7.13 | 8.16 |  |

where the fractional coefficients are =

|  |  |
|---|---|
| $ade$ + 2<br>$a^0b^2f$ + 4<br>,, $bce$ − 6<br>,, $cl$ + 4 | $a^2b^3$ + 2<br>,, $d^2$ + 6<br>$abcd$ − 2<br>$a^0b^2c^2$ − 8<br>,, $bef$ − 3<br>,, $c^2h$ + 1<br>,, $ce^2$ + 1 |

I divided and segregate is

|  |  |  |  |  | ÷3 | ÷3 | ÷3 |  |
|---|---|---|---|---|---|---|---|---|
| $I = a^3($ | $i$ + 1 | $ab^2$ − 2<br>,, $h$ + 2<br>$a^0cd$ −18 | $al$ + 5<br>$a^0ci$ −15 | $a^2bd$ −20<br>$a^0c^2d$ +60 | $a^3k$ − 5<br>$a^2bi$ −25<br>$acl$ −30<br>$a^0c^2i$ +45 | $a^4j$ − 4<br>$a^3b^3$ +10<br>,, $bh$ − 8<br>,, $cg$ + 4<br>$a^2bcd$ − 6<br>$abc^2$ −18<br>,, $c^2h$ − 6<br>$a^0c^3d$ −54 | $a^3b^2e$ + 3<br>,, $bl$ + 9<br>,, $ck$ − 5<br>,, $fg$ − 3<br>$a^2bci$ − 8<br>$ab^2cf$ − 9<br>,, $bc^2e$ −15<br>,, $c^2l$ + 3<br>$a^0c^3i$ − 3 | $)(x, y)^6,$ |
|  | 4.6 | 5.9 | 6.12 | 7.15 | 8.18 | 9.21 | 10.24 |  |

where the fractional coefficients are=

|  |  |  |
|---|---|---|
| $a^2de$ − 5<br>$ab^2j$ + 5<br>,, $bce$ − 5<br>$a^0c^2i$ − 5<br>,, $cdf$ +30 | $a^3b^3$ + 2<br>,, $d^2$ −12<br>$a^2bcd$ − 2<br>$ab^2c^2$ − 6<br>,, $c^2h$ − 2<br>,, $ce^2$ − 2<br>$a^0c^3d$ −18 | $a^3b^2e$ + 1<br>$a^2bdf$ + 3<br>,, $cde$ − 5<br>$ab^2cf$ + 1<br>,, $bc^2e$ − 5<br>,, $e^2f$ − 1<br>$a^0c^3i$ + 1<br>,, $c^2df$ − 3 |

Table No. 98 (continued).

J divided and segregate is

$$\mathrm{J} = a^7\,(j,\ -n)(x,\ y)^1.$$

5.6 6.4

K divided and segregate is

| | | | | | |
|---|---|---|---|---|---|
| $\mathrm{K}=a^6($ | $k$ + 1 | $aj$ − 2<br>$a^0b^3$ + 6<br>,, $bh$ − 9<br>,, $cg$ + 3 | $an$ + 1<br>$a^0ck$ − 3 | $a^2m$ − 3<br>$acj$ + 1<br>$a^0b^3c$ − 2<br>,, $bch$ + 3<br>,, $c^2g$ − 1 | $)(x,\ y)^3.$ |
| | 5.3 | 6.6 | 7.9 | 8.12 | |

L divided and (as to first six coefficients) segregate is

| | | | | ÷3 | ÷3 | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $\mathrm{L}=a^4($ | $l$ + 1 | $abd$ − 3<br>$a^0b^2c$ − 7<br>,, $ch$ + 7 | $a^2k$ − 4<br>$abi$ + 1<br>$a^0cl$ −21 | $a^3j$ − 13<br>$a^2b^3$ − 5<br>,, $bh$ − 5<br>,, $cg$ + 10<br>$abcd$ + 30<br>$a^0b^2c^2$ +105<br>,, $ch$ −105 | $a^3n$ − 3<br>$a^2bl$ − 45<br>,, $ck$ − 20<br>$abci$ − 10<br>$a^0c^2l$ + 105 | $a^4m$ − 1<br>$a^3b^2d$ +13<br>,, $cj$ + 3<br>$a^2b^3c$ −13<br>,, $bch$ +29<br>,, $c^2g$ −16<br>$abc^2d$ − 3<br>$a^0b^2c^3$ −21<br>,, $c^3h$ +21 | $a^3bde$ − 7<br>$a^2b^3f$ − 7<br>,, $b^2ce$ +14<br>,, $d^2f$ +12<br>$abcdf$ +23<br>,, $c^2de$ −24<br>$a^0b^2cf$ +25<br>,, $bc^3e$ −55<br>,, $c^3l$ − 3<br>,, $c^2fh$ + 2<br>,, $ce^2f$ − 7 | $a^4b^4$ − 2<br>,, $bd^2$ + 3<br>$a^3b^2cd$ +10<br>,, $de^2$ + 2<br>$a^2b^3c^2$ +13<br>,, $b^2ef$ + 4<br>,, $bce^2$ − 2<br>,, $c^2d^2$ −15<br>$abc^3d$ −28<br>,, $cdef$ − 7<br>$a^0b^2c^4$ −19<br>,, $bc^2ef$ −10<br>,, $c^4h$ − 1<br>,, $c^3e^2$ + 5 | $)(x,\ y)^7$ |
| | 5.7 | 6.10 | 7.13 | 8.16 | 9.19 | 10.22 | 11.25 | 12.28 | |

where the fractional coefficients are =

| | |
|---|---|
| $a^2b^3$ − 6<br>,, $d^2$ + 3<br>$abcd$ +26<br>$a^0b^2c^2$ +31<br>,, $bef$ + 7<br>,, $c^2h$ − 7<br>,, $ce^2$ − 1<br>,, $fl$ −14 | $a^2b^2e$ − 1<br>$abdf$ +39<br>,, $cde$ −22<br>$a^0b^2cf$ +16<br>,, $bc^2e$ − 4<br>,, $c^2l$ +19<br>,, $cfh$ − 8<br>,, $e^2f$ + 1 |

the last two coefficients have not been reduced to the segregate form.

M divided and segregate is

| | | | | |
|---|---|---|---|---|
| $\mathrm{M}=a^8($ | $m$ + 1 | $bk$ − 1<br>$p$ − 1 | $abj$ − 1<br>$adg$ + 1<br>,,$^0cm$ − 1 | $)(x,\ y)^2.$ |
| | 6.2 | 7.5 | 8.8 | |

Table No. 98 (continued).

N divided and segregate is

| | | | | | | |
|---|---|---|---|---|---|---|
| $N=a^7($ | $n + 1$ | $cj + 4$ | $ap - 6$<br>$cn - 6$ | $a^2bj - 4$<br>,, $dg + 4$<br>$ab^4 + 8$<br>,, $b^2h - 12$<br>,, $bcg + 4$<br>,, $cm - 8$<br>$a^0c^2j - 4$ | $a^3o + 1$<br>$a^2bn - 1$<br>$acp + 2$<br>$a^0c^2n + 1$ | $)\!(x, y)^4.$ |
| | 6.4 | 7.7 | 8.10 | 9.13 | 10.16 | |

O divided and segregate is

| | | | |
|---|---|---|---|
| $O=a^{10}($ | $o + 1$ | $b^2g + 1$<br>$bm + 6$<br>$df - 6$<br>$gh - 1$ | $)\!(x, y)^1.$ |
| | 7.1 | 8.4 | |

P divided and (as to first three coefficients) segregate is

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $P=a^8($ | $p + 1$ | $abj + 8$<br>,, $dg - 5$<br>$a^0b^4 - 14$<br>,, $b^2h + 15$<br>,, $bcg - 5$<br>,, $cm + 10$ | $a^2o + 7$<br>$abn - 2$<br>$a^0cp - 14$ | $a^2bm + 8$<br>,, $dj + 3$<br>$ab^3d + 9$<br>,, $bcj + 13$<br>,, $bdh - 9$<br>,, $cdg - 12$<br>,, $d^3 - 81$<br>$a^0b^4c - 5$<br>,, $b^2ch - 8$<br>,, $b^2e^2 - 1$<br>,, $bc^2g - 1$<br>,, $bcd^2 - 9$<br>,, $bel - 1$<br>,, $bfk + 1$<br>,, $c^2m - 12$<br>,, $dei + 9$<br>,, $e^2h + 1$<br>,, $fp - 2$ | $a^2b^2k - 3$<br>,, $bp - 3$<br>,, $em + 2$<br>$ab^2de + 3$<br>,, $bdl + 3$<br>,, $bfj + 8$<br>,, $d^2i - 27$<br>,, $deh - 3$<br>,, $dfg - 8$<br>$a^0b^3ce - 7$<br>,, $b^2cl + 1$<br>,, $bc^2k + 9$<br>,, $bcdi - 9$<br>,, $bceh + 10$<br>,, $c^2eg - 3$<br>,, $c^2p + 9$<br>,, $cd^2e + 18$<br>,, $cfm + 4$<br>,, $efk + 3$ | $a^3b^2j + 2$<br>,, $bdg - 2$<br>$a^2bcm + 2$<br>,, $bek - 1$<br>,, $ep - 1$<br>$ab^3cd + 24$<br>,, $bc^2j - 6$<br>,, $bcdh - 33$<br>,, $c^2dg + 15$<br>,, $cd^3 - 54$<br>,, $dfk - 9$<br>$a^0b^4c^3 + 8$<br>,, $b^2c^2h - 11$<br>,, $bc^3g + 3$<br>,, $bc^2d^2 - 18$<br>,, $bcfk - 1$<br>,, $c^3m - 6$<br>,, $cfp + 2$ | $)\!(x, y)^5,$ |
| | 7.5 | 8.8 | 9.11 | 10.14 | 11.17 | 10.20 | |

the last three coefficients have not been reduced to the segregate form.

Q as an invariant was divided and segregate, $Q=a^{12}\ \underset{8.0}{q}$

Table No. 98 (continued).

R divided and segregate is

$\mathrm{R}=a^{11}($

| | | | ÷2 | | | ÷3 | | |
|---|---|---|---|---|---|---|---|---|
| $r$ | + | 1 | $aq$ | + | 1 | $abo$ | − | 1 |
| | | | $a^0b^2j$ | + | 6 | ,, $gk$ | − | 1 |
| | | | ,, $bdg$ | − | 5 | ,, $s$ | + | 3 |
| | | | ,, $hj$ | − | 1 | $a^0cr$ | − | 3 |
| 8.2 | | | 9.5 | | | 10.8 | | |

$)(x, y)^2$

where the fractional coefficients are =

| | | | | | |
|---|---|---|---|---|---|
| $b^2j$ | + | 2 | $bdk$ | + | 3 |
| $bdg$ | − | 2 | $bej$ | + | 1 |
| $dm$ | − | 6 | $cr$ | + | 1 |
| | | | $deg$ | − | 1 |
| | | | $dp$ | + | 3 |

S divided and (as to the first three coefficients) segregate is

$\mathrm{S}=a^{12}($

| | | | | | | | | | ÷ 2 | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $s$ | + | 1 | $agj$ | − | 2 | $abr$ | − | 1 | $a^2bq$ | + | 4 |
| | | | $a^0b^3g$ | + | 2 | ,, $do$ | − | 1 | $ab^3j$ | + | 4 |
| | | | ,, $b^2m$ | + | 3 | $a^0cs$ | − | 3 | ,, $b^2dg$ | − | 4 |
| | | | ,, $bdj$ | + | 21 | | | | ,, $bdm$ | − | 31 |
| | | | ,, $bgh$ | − | 4 | | | | ,, $d^2j$ | − | 3 |
| | | | ,, $cg^2$ | + | 2 | | | | $a^0b^6$ | + | 4 |
| | | | ,, $cq$ | − | 3 | | | | ,, $b^3d^2$ | + | 16 |
| | | | | | | | | | ,, $bd^2h$ | − | 24 |
| | | | | | | | | | ,, $den$ | + | 4 |
| | | | | | | | | | ,, $fs$ | − | 1 |
| 9.3 | | | 10.6 | | | 11.9 | | | 12.12 | | |

$)(x, y)^3$,

but the last coefficient is neither segregate nor integer.

T divided and segregate is

$\mathrm{T}=a^{16}($

| | | | ÷2 | | |
|---|---|---|---|---|---|
| $t$ | + | 1 | $bgm$ | + | 1 |
| | | | $bj^2$ | + | 4 |
| | | | $dgj$ | − | 3 |
| | | | $hq$ | − | 1 |
| 11.1 | | | 12.4 | | |

$)(x, y)^1$,

where the fractional coefficient is =

| | | |
|---|---|---|
| $b^2q$ | − | 1 |
| $hq$ | + | 1 |
| $m^2$ | + | 6 |

Table No. 98 (concluded).

U as an invariant was divided and segregate, $\mathrm{U}=a^{18}\underset{12.0}{u}$

V divided and segregate is

$$\mathrm{V}=a^{19}\Bigg(\begin{array}{|ccc|ccc|} \multicolumn{3}{c}{} & \multicolumn{3}{r}{\div 6} \\ \hline v & + & 1 & bgr & - & 5 \\ & & & bjo & - & 10 \\ & & & gjk & + & 5 \\ & & & js & - & 12 \\ & & & nq & - & 9 \\ \hline \multicolumn{3}{c}{13.1} & \multicolumn{3}{c}{14.4} \end{array}$$

where the fractional coefficient is =

$$\begin{array}{|ccc|} \hline dt & - & 6 \\ mr & - & 6 \\ nq & + & 1 \\ \hline \end{array}$$

W as an invariant was divided and segregate, $\mathrm{W}=a^{27}\underset{18.0}{w}$

## *Derivatives.*—Article Nos. 382 to 384, and Tables Nos. 99 and 100.

382. I call to mind that any two covariants $a$, $b$, the same or different, give rise to a set of derivatives $(a, b)^1$, $(a, b)^2$, $(a, b)^3$, &c., or, as I propose to write them, $ab1$, $ab2$, $ab3$, &c., viz. :

$$\begin{aligned} ab1&=d_x a \,.\, d_y b-d_y a \,.\, d_x b \\ ab2&=d_x^2 a \,.\, d_y^2 b-2d_x d_y a \,.\, d_x d_y b+d_y^2 a \,.\, d_x^2 b \\ ab3&=d_x^3 a \,.\, d_y^3 b-3d_x^2 d_y a \,.\, d_x d_y^2 b+3d_x d_y^2 a \,.\, d_x^2 d_y b-d_y^3 a \,.\, d_x^3 b \\ &\text{\&c.} \end{aligned}$$

or, as these are symbolically written, $ab1=\overline{12}a_1b_2$, $ab2=\overline{12}^2a_1b_2$, $ab3=\overline{12}^3a_1b_2$, &c.; where $12=\xi_1\eta_2-\xi_2\eta_1$, $=\frac{d}{dx_1}\frac{d}{dy_2}-\frac{d}{dx_2}\frac{d}{dy_1}$, the differentiations $\frac{d}{dx_1}$, $\frac{d}{dy_1}$ applying to the $a_1$ and the $\frac{d}{dx_2}$, $\frac{d}{dy_2}$ applying to the $b_2$, but the suffixes being ultimately omitted : hence if $\theta$ be the index of derivation, the derivative is thus a linear function of the differential coefficients of the order $\theta$ of the two covariants $a$ and $b$ respectively : and we have the general property that any such derivative, if not identically vanishing, is a covariant. If the $a$ and the $b$ are one and the same covariant, then obviously every odd derivative is $=0$; so that in this case the only derivatives to be considered are the even derivatives $aa2$, $aa4$, &c. : moreover, if the index of derivation $\theta$ exceeds the order of either of the component covariants, then also the derivative is $=0$ : in particular neither of the covariants must be an invariant. The degree of the derivative is evidently equal to the sum of the degrees of the component covariants; the order is equal to the sum of the orders less twice the index of derivation.

383. It was by means of the theory of derivatives that GORDAN proved (for a binary quantic of any order) that the number of covariants was finite, and in the particular case of the quintic, established the system of the 23 covariants. Starting from the quantic itself $a$, then the system of derivatives $aa2$, $aa4$, &c., must include among itself all the covariants of the second degree, and if the entire system of these is, suppose, $b$, $c$, then the derivatives $ab1$, $ab2$, $ab3$, &c., $ac1$, $ac2$, &c., must include among them all the covariants of the third degree, and so on for the higher degrees; and in this way limiting by general reasoning the number of the independent covariants of each degree obtained by the successive steps, the foregoing conclusion is arrived at. But returning to the quintic, and supposing the system of the 23 covariants established, then knowing the deg-order of a derivative we know that it must be a linear function of the segregates of that deg-order; and we thus confirm, *à posteriori*, the results of the derivation theory. I annex the following Table No. 99, which shows all the derivatives which present themselves, and for each of them the covariants as well congregate as segregate of the same deg-order: the congregates are distinguished each by two prefixed dots, . . $bf$, &c. No further explanation of the arrangement is, I think, required. We see from the table in what manner the different covariants present themselves in connexion with the derivation-theory. Thus starting with the quintic itself $a$, we have the two derivatives $aa4$, $aa2$, which are in fact the covariants of the second degree (deg-orders 2.2 and 2.6 respectively) $b$ and $c$. For the third degree we have the derivatives $ab2$, $ab1$, $ac5$, $ac4$, $ac3$, $ac2$, $ac1$ : the deg-order of $ac5$ is 3.1, and there being no covariants of this deg-order, $ac5$ must, it is clear, vanish identically : $ab2$ and $ac4$ are each of them of the deg-order 3.3, but for this deg-order we have only the covariant $d$, and hence $ab2$ and $ac4$ must be each of them a numerical multiple of $d$; similarly, deg-order 3.5, $ab1$ and $ac3$ must be each of them a numerical multiple of $e$; deg-order 3.7, $ac2$ must be a numerical multiple of $ab$; and deg-order 3.9, $ac1$ must be a numerical multiple of $f$: the 7 derivatives, which *primâ facie* might give, each of them, a covariant of the third degree, thus give in fact only the 3 covariants $d$, $e$, $f$; and in order to show according to the theory of derivations that this is so, it is necessary to prove—1°, that $ac5=0$; 2°, that $ac4$ and $ab2$ differ only by a numerical factor; 3°, that $ab1$ and $ac3$ differ only by a numerical factor; 4°, that $ac2$ is a numerical multiple of $ab$ : which being so we have the 3 new covariants. The table shows that

| for degrees | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| No. of derivatives= | 2 | 7 | 19 | 29 | 41 | 46 | 52 | 46 | 44 | 35 | 26 | 19 | 17 | 12 | 13 | 6 | 6 | 3 | 3 | 1 | 1 | 0 | 1 |

so that the whole number of derivatives is 429, giving the 22 covariants $b$, $c$ . . . $w$.

While it is very remarkable that (by general reasoning, as already mentioned, and with a very small amount of calculation) GORDAN should have been able in effect to show this, the great excess of the number of derivatives over that of the covariants seems a reason why the derivations ought not to be made a basis of the theory.

It is to be remarked that we may consider derivatives $pq1$, $pq2$, &c., where $p$, $q$ instead of being simple covariants are powers or products of covariants; but that these may be made to depend upon the derivatives formed with the simple covariants. (As to this see my paper "On the Derivatives of Three Binary Quantics," Quart. Math. Journal, t. xv. (1877) pp. 157–168.)

Table No. 99 (Index Table of Derivatives).

| Deg. | 2 | | | | |
|---|---|---|---|---|---|
| Ord. | | 0 | 2 | 4 | 6 |
| | | | *b* | | *c* |
| | *aa* | | 4 | | 2 |

2 derivs.

| Deg. | 3 | | | | | |
|---|---|---|---|---|---|---|
| Ord. | | 1 | 3 | 5 | 7 | 9 |
| | | | *d* | *e* | *ab* | *f* |
| | *ab* | | 2 | 1 | | |
| | *ac* | 5 | 4 | 3 | 2 | 1 |

7 derivs.

| Deg. | 4 | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Ord. | | 0 | 2 | 4 | 6 | 8 | 10 | 12 |
| | | *g* | | $b^2$<br>*h* | *i* | *ad*<br>*bc* | *ae* | $a^2b$<br>$c^2$ |
| | *ad* | | 3 | 2 | 1 | | | |
| | *ae* | 5 | 4 | 3 | 2 | 1 | | |
| | *af* | | | 5 | 4 | 3 | 2 | 1 |
| | *bb* | 2 | | | | | | |
| | *bc* | | | 2 | 1 | | | |
| | *cc* | 6 | | 4 | | 2 | | |

19 derivs.

| Deg. | 5 | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Ord. | | 1 | 3 | 5 | 7 | 9 | 11 | 13 |
| | | *j* | *k* | *ag*<br>*bd* | *be*<br>*l* | $ab^2$<br>*ah*<br>*cd* | *ai*<br>. . *bf*<br>*ce* | $a^2d$<br>*abc* |
| | *ah* | 4 | 3 | 2 | 1 | | | |
| | *ai* | 5 | 4 | 3 | 2 | 1 | | |
| | *bd* | 2 | 1 | | | | | |
| | *be* | | 2 | 1 | | | | |
| | *bf* | | | | 2 | 1 | | |
| | *cd* | | 3 | 2 | 1 | | | |
| | *ce* | 5 | 4 | 3 | 2 | 1 | | |
| | *cf* | | 6 | 5 | 4 | 3 | 2 | 1 |

29 derivs.

Table No. 99 (continued).

| Deg. | 6 | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Ord. | 0 | 2 | 4 | 6 | 8 | 10 | 12 | 14 |
| | | *bg*<br>*m* | *n* | *aj*<br>$b^3$<br>*bh*<br>*cg*<br>. . $d^3$ | *ak*<br>*bi*<br>. . *de* | $a^2g$<br>*abd*<br>$b^2c$<br>*ch*<br>. . $e^2$ | *abe*<br>*al*<br>*ci*<br>. . *df* | $a^2b^2$<br>$a^2h$<br>*acd*<br>$bc^2$<br>. . *ef* |
| *aj* | | | 1 | | | | | |
| *ak* | | 3 | 2 | 1 | | | | |
| *al* | | 5 | 4 | 3 | 2 | 1 | | |
| *bh* | | 2 | 1 | | | | | |
| *bi* | | | 2 | 1 | | | | |
| *ch* | | 4 | 3 | 2 | 1 | | | |
| *ci* | 6 | 5 | 4 | 3 | 2 | 1 | | |
| *dd* | | 2 | | | | | | |
| *de* | | 3 | 2 | 1 | | | | |
| *df* | | | | 3 | 2 | 1 | | |
| *ee* | | 4 | | 2 | | | | |
| *ef* | | | 5 | 4 | 3 | 2 | 1 | |
| *ff* | | 8 | | 6 | | 4 | | 2 |

41 derivs.

| Deg. | 7 | | | | | | |
|---|---|---|---|---|---|---|---|
| Ord. | 1 | 3 | 5 | 7 | 9 | 11 | 13 |
| | *o* | *bj*<br>*dg* | *bk*<br>*eg*<br>*p* | *abg*<br>*am*<br>$b^2d$<br>*cj*<br>. . *dh* | *an*<br>. . $b^2e$<br>*bl*<br>*ck*<br>. . *di*<br>. . *eh*<br>*fg* | $a^2j$<br>$ab^3$<br>*abh*<br>*acg*<br>. . $ad^2$<br>*bcd*<br>. . *ei* | $a^2k$<br>*abi*<br>. . *ade*<br>. . $b^2f$<br>*bce*<br>*cl*<br>. . *fh* |
| *am* | | 2 | 1 | | | | |
| *an* | 4 | 3 | 2 | 1 | | | |
| *bj* | 1 | | | | | | |
| *bk* | 2 | 1 | | | | | |
| *bl* | | | 2 | 1 | | | |
| *cj* | | | 1 | | | | |
| *ck* | | 3 | 2 | 1 | | | |
| *cl* | 6 | 5 | 4 | 3 | 2 | 1 | |
| *dh* | 3 | 2 | 1 | | | | |
| *di* | | 3 | 2 | 1 | | | |
| *eh* | 4 | 3 | 2 | 1 | | | |
| *ei* | 5 | 4 | 3 | 2 | 1 | | |
| *fh* | | | 4 | 3 | 2 | 1 | |
| *fi* | | 6 | 5 | 4 | 3 | 2 | 1 |

46 derivs.

Table No. 99 (continued).

| Deg. | 8 | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Ord. | 0 | 2 | 4 | 6 | 8 | 10 | 12 | 14 |
| | $g^2$<br>$q$ | $r$ | $b^2g$<br>$bm$<br>$dj$<br>$gh$ | $as$<br>$bn$<br>. . $dk$<br>. . $ej$<br>$gi$ | $abj$<br>$adg$<br>$b^4$<br>$b^2h$<br>$bcg$<br>. . $bd^2$<br>$cm$<br>. . $ek$<br>. . $h^2$ | $abk$<br>$aeg$<br>$ap$<br>$b^2i$<br>. . $bde$<br>$cn$<br>. . $dl$<br>. . $fj$<br>. . $hi$ | $a^2bg$<br>$a^2m$<br>$ab^2d$<br>$acj$<br>. . $adh$<br>$b^3c$<br>$bch$<br>. . $be^2$<br>$c^2g$<br>. . $cd^2$<br>. . $el$<br>. . $fk$<br>. . $i^2$ | $a^2n$<br>. . $ab^2e$<br>$abl$<br>$ack$<br>. . $adi$<br>. . $aeh$<br>$afg$<br>$bci$<br>. . $bdf$<br>. . $cde$ |
| $ao$ | | | 1 | | | | | |
| $ap$ | 5 | 4 | 3 | 2 | 1 | | | |
| $bm$ | 2 | 1 | | | | | | |
| $bn$ | | 2 | 1 | | | | | |
| $cm$ | | | 2 | 1 | | | | |
| $cn$ | | 4 | 3 | 2 | 1 | | | |
| $dj$ | | 1 | | | | | | |
| $dk$ | 3 | 2 | 1 | | | | | |
| $dl$ | | | 3 | 2 | 1 | | | |
| $ej$ | | | 1 | | | | | |
| $ek$ | | 3 | 2 | 1 | | | | |
| $el$ | | 5 | 4 | 3 | 2 | 1 | | |
| $fj$ | | | | | 1 | | | |
| $fk$ | | | | 3 | 2 | 1 | | |
| $fl$ | | 7 | 6 | 5 | 4 | 3 | 2 | 1 |
| $hh$ | 4 | | 2 | | | | | |
| $hi$ | | 4 | 3 | 2 | 1 | | | |
| $ii$ | 6 | | 4 | | 2 | | | |

52 derivs.

Table No. 99 (continued).

| Deg. | 9 | | | | | |
|---|---|---|---|---|---|---|
| Ord. | 1 | 3 | 5 | 7 | 9 | 11 |
| | $gj$ | $bo$<br>$gk$<br>$s$ | $ag^2$<br>$aq$<br>$b^2j$<br>$bdg$<br>. . $dm$<br>$hj$ | $ar$<br>. . $b^2k$<br>$beg$<br>$bp$<br>$co$<br>. . $dn$<br>. . $em$<br>$gl$<br>. . $hk$<br>. . $ij$ | $ab^2g$<br>$abm$<br>$adj$<br>$agh$<br>$b^3d$<br>$bcj$<br>. . $bdh$<br>$cdg$<br>. . $d^3$<br>. . $en$<br>. . $ik$ | $a^2o$<br>$abn$<br>. . $adk$<br>. . $aej$<br>$agi$<br>. $b^3e$<br>$b^2l$<br>$bck$<br>. . $bdi$<br>. . $beh$<br>. . $bfg$<br>$ceg$<br>$cp$<br>. . $d^2e$<br>. . $fm$<br>. . $hl$ |
| $ar$ | | 2 | 1 | | | |
| $bo$ | 1 | | | | | |
| $bp$ | | 2 | 1 | | | |
| $co$ | | | 1 | | | |
| $cp$ | 5 | 4 | 3 | 2 | 1 | |
| $dm$ | 2 | 1 | | | | |
| $dn$ | 3 | 2 | 1 | | | |
| $em$ | | 2 | 1 | | | |
| $en$ | 4 | 3 | 2 | 1 | | |
| $fm$ | | | | 2 | 1 | |
| $fn$ | | | 4 | 3 | 2 | 1 |
| $hj$ | | 1 | | | | |
| $hk$ | 3 | 2 | 1 | | | |
| $hl$ | | 4 | 3 | 2 | 1 | |
| $ij$ | | | 1 | | | |
| $ik$ | | 3 | 2 | 1 | | |
| $il$ | 6 | 5 | 4 | 3 | 2 | 1 |

46 derivs.

## Table No. 99 (continued).

| Deg. | 10 | | | | | | |
|---|---|---|---|---|---|---|---|
| Ord. | 0 | 2 | 4 | 6 | 8 | 10 | 12 |
| | | $bg^2$<br>$bq$<br>$gm$<br>$j^2$ | $br$<br>. . $do$<br>$gn$<br>$jk$ | $agj$<br>$b^3g$<br>$b^2m$<br>$bdj$<br>$bgh$<br>$cg^2$<br>$cq$<br>. . $d^2g$<br>. . $eo$<br>. . $hm$<br>. . $k^2$ | $abo$<br>$agk$<br>$as$<br>$b^2n$<br>. . $bdk$<br>. . $bej$<br>$bgi$<br>$cr$<br>. . $deg$<br>. . $dp$<br>. . $hn$<br>. . $im$<br>. . $jl$ | $a^2g^2$<br>$a^2q$<br>$ab^2j$<br>$abdg$<br>. . $adm$<br>$ahj$<br>. . $b^5$<br>$b^3h$<br>$b^2cg$<br>. . $b^2d^2$<br>$bcm$<br>. . $bek$<br>. . $bh^2$<br>$cdj$<br>$cgh$<br>. . $d^2h$<br>. . $e^2g$<br>. . $ep$<br>. . $fo$<br>. . $in$<br>. . $kl$ | $a^2r$<br>. . $ab^2k$<br>$abeg$<br>$abp$<br>$aco$<br>. . $adn$<br>. . $aem$<br>$agl$<br>. . $ahk$<br>. . $aij$<br>$b^3i$<br>. . $b^2de$<br>$bcn$<br>. . $bdl$<br>. . $bfj$<br>. . $bhi$<br>. . $cdk$<br>. . $cej$<br>$cgi$<br>. . $d^2i$<br>. . $deh$<br>. . $dfg$ |
| $as$ | | 3 | 2 | 1 | | | |
| $br$ | 2 | 1 | | | | | |
| $cr$ | | | 2 | 1 | | | |
| $do$ | | 1 | | | | | |
| $dp$ | | 3 | 2 | 1 | | | |
| $eo$ | | | 1 | | | | |
| $ep$ | 5 | 4 | 3 | 2 | 1 | | |
| $fo$ | | | | | 1 | | |
| $fp$ | | | 5 | 4 | 3 | 2 | 1 |
| $hm$ | | 2 | 1 | | | | |
| $hn$ | 4 | 3 | 2 | 1 | | | |
| $im$ | | | 2 | 1 | | | |
| $in$ | | 4 | 3 | 2 | 1 | | |
| $jk$ | | 1 | | | | | |
| $jl$ | | | | 1 | | | |
| $kk$ | | 2 | | | | | |
| $kl$ | | | 3 | 2 | 1 | | |
| $ll$ | | 6 | | 4 | | 2 | |

44 derivs.

Table No. 99 (continued).

| Deg. | 11 | | | | |
|---|---|---|---|---|---|
| Ord. | 1 | 3 | 5 | 7 | 9 |
| | go | bgj | $b^2o$ | $abg^2$ | abr |
| | t | $dg^2$ | bgk | abq | . . ado |
| | | dq | bs | agm | agn |
| | | jm | . . dr | $aj^2$ | ajk |
| | | | $eg^2$ | $b^3j$ | . . $b^3k$ |
| | | | eq | $b^2dg$ | . . $b^2eg$ |
| | | | gp | . . bdm | $b^2p$ |
| | | | . . ho | bhj | bco |
| | | | . . jn | cgj | . . bdn |
| | | | . . km | . . $d^2j$ | . . bem |
| | | | | . . dgh | bgl |
| | | | | . . er | . . bhk |
| | | | | . . io | . . bij |
| | | | | . . kn | cjk |
| | | | | | cs |
| | | | | | . . $d^2k$ |
| | | | | | . . dej |
| | | | | | . . dgi |
| | | | | | . . egh |
| | | | | | $fg^2$ |
| | | | | | fq |
| | | | | | . . hp |
| | | | | | . . lm |
| bs | 2 | 1 | | | |
| cr | | 3 | 2 | 1 | |
| dr | 2 | 1 | | | |
| er | | 2 | 1 | | |
| fr | | | | 2 | 1 |
| ho | | 1 | | | |
| hp | 4 | 3 | 2 | 1 | |
| io | | | 1 | | |
| ip | 5 | 4 | 3 | 2 | 1 |
| jm | 1 | | | | |
| jn | | 1 | | | |
| km | 2 | 1 | | | |
| kn | 3 | 2 | 1 | | |
| lm | 2 | 1 | | | |
| ln | | 4 | 3 | 2 | 1 |

35 derivs.

Table No. 99 (continued).

| Deg. | 12 | | | | | |
|---|---|---|---|---|---|---|
| Ord. | 0 | 2 | 4 | 6 | 8 | 10 |
| | $g^3$<br>$g^9$<br>$u$ | $gr$<br>$jo$ | $b^2g^2$<br>$b^2q$<br>$bgm$<br>$bj^2$<br>$dgj$<br>$g^2h$<br>$hq$<br>. . $ko$<br>. . $m^2$ | $ago$<br>$at$<br>$b^2r$<br>. . $bdo$<br>$bgn$<br>$bjk$<br>. . $dgk$<br>. . $ds$<br>. . $egj$<br>$g^2i$<br>. . $hr$<br>$iq$<br>. . $jp$<br>. . $mn$ | $abgj$<br>. . $adg^2$<br>$adq$<br>$ajm$<br>$b^4g$<br>$b^3m$<br>$b^2dj$<br>$b^2gh$<br>$bcg^2$<br>$bcq$<br>. . $bd^2g$<br>. . $beo$<br>. . $bhm$<br>. . $bk^2$<br>$cgm$<br>$cj^2$<br>. . $d^2m$<br>. . $dhj$<br>. . $egk$<br>. . $es$<br>. . $gh^2$<br>. . $ir$<br>. . $kp$<br>. . $lo$<br>. . $n^2$ | $ab^2o$<br>$abgk$<br>$abs$<br>. . $adr$<br>$aeg^2$<br>$aeq$<br>$agp$<br>. . $aho$<br>. . $ajn$<br>. . $akm$<br>$b^3n$<br>. . $b^2dk$<br>. . $b^2ej$<br>. . $b^2gi$<br>$bcr$<br>. . $bdeg$<br>. . $bdp$<br>. . $bhn$<br>. . $bim$<br>. . $bjl$<br>. . $cdo$<br>$cgn$<br>$cjk$<br>. . $d^2n$<br>. . $dem$<br>. . $dgl$<br>. . $dhk$<br>. . $dij$<br>. . $ehj$<br>. . $fgj$<br>. . $ghi$ |
| $at$ | | | 1 | | | |
| $ds$ | 3 | 2 | 1 | | | |
| $es$ | | 3 | 2 | 1 | | |
| $fs$ | | | | 3 | 2 | 1 |
| $hr$ | | 2 | 1 | | | |
| $ir$ | | | 2 | 1 | | |
| $jo$ | 1 | | | | | |
| $jp$ | | | 1 | | | |
| $ko$ | | 1 | | | | |
| $kp$ | | 3 | 2 | 1 | | |
| $lo$ | | | | 1 | | |
| $lp$ | | 5 | 4 | 3 | 2 | 1 |

26 derivs.

Table No. 99 (continued).

| Deg. | 13 | | | |
|---|---|---|---|---|
| Ord. | 1 | 3 | 5 | 7 |
| | $g^2j$ | $bgo$ | $ag^3$ | $agr$ |
| | $jq$ | $bt$ | $agq$ | $ajo$ |
| | $v$ | $g^2k$ | $au$ | . . $b^3o$ |
| | | $gs$ | $b^2gj$ | . . $b^2gk$ |
| | | . . $jr$ | $bdg^2$ | $b^2s$ |
| | | $kq$ | $bdq$ | . . $bdr$ |
| | | . . $mo$ | $bjm$ | $beg^2$ |
| | | | . . $dgm$ | $beq$ |
| | | | . . $dj^2$ | $bgp$ |
| | | | $ghj$ | . . $bho$ |
| | | | . . $kr$ | . . $bjn$ |
| | | | . . $no$ | . . $bkm$ |
| | | | | $cgo$ |
| | | | | $ct$ |
| | | | | . . $d^2o$ |
| | | | | . . $dgn$ |
| | | | | . . $djk$ |
| | | | | . . $egm$ |
| | | | | . . $ej^2$ |
| | | | | $g^2l$ |
| | | | | . . $ghk$ |
| | | | | . . $gij$ |
| | | | | . . $hs$ |
| | | | | . . $lq$ |
| | | | | . . $mp$ |
| $bt$ | 1 | | | |
| $ct$ | | | 1 | |
| $hs$ | 3 | 2 | 1 | |
| $is$ | | 3 | 2 | 1 |
| $jr$ | 1 | | | |
| $kr$ | | 1 | | |
| $lr$ | | | | 1 |
| $mo$ | 1 | | | |
| $mp$ | | 2 | 1 | |
| $no$ | | 1 | | |
| $np$ | 4 | 3 | 2 | 1 |

19 derivs.

Table No. 99 (continued).

| Deg. | 14 | | | | |
|---|---|---|---|---|---|
| Ord. | 0 | 2 | 4 | 6 | 8 |
| | | $bg^3$ | $bgr$ | . . $ag^2j$ | $abgo$ |
| | | $bgq$ | $bjo$ | $ajq$ | $abt$ |
| | | $bu$ | . . $dgo$ | $av$ | $ag^2k$ |
| | | $g^2m$ | . . $dt$ | $b^3g^2$ | $ags$ |
| | | $gj^2$ | $g^2n$ | $b^3q$ | . . $ajr$ |
| | | $mq$ | $gjk$ | $b^2gm$ | $akq$ |
| | | . . $o^2$ | $js$ | $b^2j^2$ | . . $amo$ |
| | | | . . $mr$ | $bdgj$ | $b^3r$ |
| | | | $nq$ | $bg^2h$ | . . $b^2do$ |
| | | | | $bhq$ | . . $b^2gn$ |
| | | | | . . $bko$ | $b^2jk$ |
| | | | | . . $bm^2$ | . . $bdgk$ |
| | | | | $cg^3$ | . . $bds$ |
| | | | | $cgq$ | . . $begj$ |
| | | | | $cu$ | $bg^2i$ |
| | | | | . . $d^2g^2$ | . . $bhr$ |
| | | | | . . $d^2q$ | $biq$ |
| | | | | . . $djm$ | . . $bjp$ |
| | | | | . . $ego$ | . . $bmn$ |
| | | | | . . $et$ | $cgr$ |
| | | | | . . $ghm$ | $cjo$ |
| | | | | . . $gk^2$ | . . $d^2r$ |
| | | | | . . $hj^2$ | . . $deg^2$ |
| | | | | . . $ks$ | . . $deq$ |
| | | | | . . $nr$ | . . $dgp$ |
| | | | | . . $op$ | . . $dho$ |
| | | | | | . . $djn$ |
| | | | | | . . $dkm$ |
| | | | | | . . $ejm$ |
| | | | | | . . $ghn$ |
| | | | | | . . $gim$ |
| | | | | | . . $gjl$ |
| | | | | | . . $hjk$ |
| | | | | | . . $ij^2$ |
| $dt$ | | | 1 | | |
| $et$ | | | | 1 | |
| $ft$ | | | | | 1 |
| $js$ | | 1 | | | |
| $ks$ | 3 | 2 | 1 | | |
| $ls$ | | | 3 | 2 | 1 |
| $mr$ | 2 | 1 | | | |
| $nr$ | | 2 | 1 | | |
| $op$ | | | 1 | | |
| $pp$ | | 4 | | 2 | |

17 derivs.

| Deg. | 15 | | |
|---|---|---|---|
| Ord. | 1 | 3 | 5 |
| | $g^2o$ | $bg^2j$ | . . $b^2go$ |
| | $gt$ | $bjq$ | $b^2t$ |
| | $oq$ | $bv$ | $bg^2k$ |
| | | $dg^3$ | $bgs$ |
| | | $dgq$ | . . $bjr$ |
| | | . . $du$ | $bkq$ |
| | | $gjm$ | . . $bmo$ |
| | | . . $j^3$ | . . $dgr$ |
| | | . . $or$ | . . $djo$ |
| | | | $eg^3$ |
| | | | $egq$ |
| | | | . . $eu$ |
| | | | $g^2p$ |
| | | | . . $gho$ |
| | | | . . $gjn$ |
| | | | . . $gkm$ |
| | | | . . $ht$ |
| | | | . . $j^2k$ |
| | | | . . $ms$ |
| | | | $pq$ |
| $bv$ | 1 | | |
| $cv$ | | | 1 |
| $ht$ | | 1 | |
| $it$ | | | 1 |
| $ms$ | 2 | 1 | |
| $ns$ | 3 | 2 | 1 |
| $or$ | 1 | | |
| $pr$ | | 2 | 1 |

12 derivs.

Table No. 99 (continued).

| Deg. | 16 | | | | | |
|---|---|---|---|---|---|---|
| Ord. | 0 | 2 | 4 | 6 | 8 | Col. 8 concl. |
| | $g^4$ | $g^2r$ | $b^2g^3$ | $ag^2o$ | . . $abg^2j$ | |
| | $g^2q$ | $gjo$ | $b^2gq$ | $agt$ | $abjq$ | . . $ekq$ |
| | $gu$ | $jt$ | $b^2u$ | $aoq$ | $abv$ | . . $emo$ |
| | $q^2$ | $qr$ | $bg^2m$ | . . $b^2gr$ | . . $adg^3$ | . . $g^2h^2$ |
| | | | $bgj^2$ | $b^2jo$ | $adgq$ | . . $gir$ |
| | | | $bmq$ | . . $bdgo$ | . . $adu$ | . . $gkp$ |
| | | | . . $bo^2$ | . . $bdt$ | $agjm$ | . . $glo$ |
| | | | $dg^2j$ | $bg^2n$ | . . $aj^3$ | . . $gn^2$ |
| | | | $djq$ | $bgjk$ | . . $aor$ | . . $h^2q$ |
| | | | . . $dv$ | $bjs$ | $b^4g^2$ | . . $hko$ |
| | | | $g^3h$ | . . $bmr$ | $b^4q$ | . . $hm^2$ |
| | | | $ghq$ | . . $bnq$ | $b^3gm$ | . . $ijo$ |
| | | | . . $gko$ | . . $dg^2k$ | $b^3j^2$ | . . $jkn$ |
| | | | . . $gm^2$ | . . $dgs$ | $b^2dgj$ | . . $k^2m$ |
| | | | . . $hu$ | . . $djr$ | $b^2g^2h$ | . . $lt$ |
| | | | . . $j^2m$ | . . $dkq$ | $b^2hq$ | . . $ps$ |
| | | | . . $kt$ | . . $dmo$ | . . $b^2ko$ | |
| | | | . . $os$ | . . $eg^2j$ | . . $b^2m^2$ | |
| | | | . . $r^2$ | . . $egj$ | $bcg^3$ | |
| | | | | . . $ev$ | $bcgq$ | |
| | | | | $g^3i$ | $bcu$ | |
| | | | | . . $ghr$ | . . $bd^2g^2$ | |
| | | | | $giq$ | . . $bd^2q$ | |
| | | | | . . $gjp$ | . . $bdjm$ | |
| | | | | . . $gmn$ | . . $bego$ | |
| | | | | . . $hjo$ | . . $bet$ | |
| | | | | . . $iu$ | . . $bghm$ | |
| | | | | . . $j^2n$ | . . $bgk^2$ | |
| | | | | . . $jkm$ | . . $bhj^2$ | |
| | | | | | . . $bks$ | |
| | | | | | . . $bnr$ | |
| | | | | | . . $bop$ | |
| | | | | | $cg^2m$ | |
| | | | | | $cgj^2$ | |
| | | | | | $cmq$ | |
| | | | | | . . $co^2$ | |
| | | | | | . . $d^2gm$ | |
| | | | | | . . $d^2j^2$ | |
| | | | | | . . $dghj$ | |
| | | | | | . . $dkr$ | |
| | | | | | . . $dno$ | |
| | | | | | . . $eg^2k$ | |
| | | | | | . . $egs$ | |
| | | | | | . . $ejr$ | |
| $dv$ | | 1 | | | | |
| $ev$ | | | 1 | | | |
| $fv$ | | | | | 1 | |
| $jt$ | 1 | | | | | |
| $kt$ | | 1 | | | | |
| $lt$ | | | | 1 | | |
| $op$ | | | 1 | | | |
| $os$ | | 1 | | | | |
| $pp$ | | 4 | | 2 | | |
| $ps$ | | 3 | 2 | 1 | | |

13 derivs.

Table No. 99 (continued).

| Deg. | 17 | | | 18 | | | | 19 | |
|---|---|---|---|---|---|---|---|---|---|
| Ord. | 1 | 3 | 5 | 0 | 2 | 4 | 6 | 1 | 3 |
| | $g^3j$<br>$gjq$<br>$gv$<br>$ju$ | $bg^2o$<br>$bgt$<br>$boq$<br>$g^3k$<br>$g^2s$<br>. . $gjr$<br>$gkq$<br>. . $gmo$<br>. . $j^2o$<br>$ku$<br>. . $mt$<br>$qs$ | $ag^4$<br>$ag^2q$<br>$agu$<br>$aq^2$<br>$b^2g^2j$<br>$b^2jq$<br>$b^2v$<br>$bdg^3$<br>$bdgq$<br>$bdu$<br>$bgjm$<br>. . $bj^3$<br>. . $bor$<br>. . $dg^2m$<br>. . $dgj^2$<br>. . $dmq$<br>. . $do^2$<br>$g^2hj$<br>. . $gkr$<br>. . $gno$<br>$hjq$<br>. . $hv$<br>. . $jko$<br>. . $jm^2$<br>. . $nt$<br>. . $rs$ | $w$ | $bg^4$<br>$bg^2q$<br>$bgu$<br>$bq^2$<br>$g^3m$<br>$g^2j^2$<br>$gmq$<br>. . $go^2$<br>$j^2q$<br>. . $jv$<br>$mu$<br>. . $ot$ | $bg^2r$<br>. . $bgjo$<br>$bjt$<br>$bqr$<br>. . $dg^2o$<br>. . $dgt$<br>. . $doq$<br>$g^3n$<br>$g^2jk$<br>$gjs$<br>. . $gmr$<br>$gnq$<br>. . $j^2r$<br>$jkq$<br>. . $jmo$<br>. . $kv$<br>$nu$ | . . $ag^3j$<br>$agjq$<br>$agv$<br>$aju$<br>$b^3g^3$<br>$b^3gq$<br>$b^3u$<br>$b^2g^2m$<br>$b^2gj^2$<br>$b^2mq$<br>. . $b^2o^2$<br>$bdg^2j$<br>$bdjq$<br>. . $bdv$<br>$bg^3h$<br>$bghq$<br>. . $bgko$<br>. . $bgm^2$<br>$bhu$<br>. . $bj^2m$<br>. . $bkt$<br>. . $bos$<br>. . $br^2$<br>$cg^4$<br>$cg^2q$<br>$cgu$<br>$cq^2$<br>. . $d^2g^3$<br>. . $d^2gq$<br>. . $d^2u$<br>. . $dgjm$<br>. . $dj^3$<br>. . $dor$<br>. . $eg^2o$<br>. . $egt$<br>. . $eoq$<br>. . $g^2hm$<br>. . $g^2k^2$<br>. . $ghj^2$<br>. . $gks$<br>. . $gnr$<br>. . $gop$<br>. . $hmq$<br>. . $ho^2$<br>. . $jkr$<br>. . $jno$<br>. . $k^2q$<br>. . $kmo$<br>. . $m^3$<br>. . $pt$<br>. . $s^2$ | $g^3o$<br>$g^2t$<br>$gqo$<br>$ou$<br>$qt$ | $bg^3j$<br>$bgjq$<br>$bgv$<br>$bju$<br>$dg^4$<br>$dg^2q$<br>. . $djt$<br>$dq^2$<br>$g^2jm$<br>. . $gj^3$<br>. . $gor$<br>$jmq$<br>. . $jo^2$<br>. . $mv$<br>. . $rt$ |
| $hv$ | | 1 | | | | | | | |
| $iv$ | | | 1 | | | | | | |
| $mt$ | 1 | | | | | | | | |
| $nt$ | | 1 | | | | | | | |
| $rs$ | 2 | 1 | | | | | | | |
| $jv$ | | | | 1 | | | | | |
| $kv$ | | | | | 1 | | | | |
| $lv$ | | | | | | | 1 | | |
| $ot$ | | | | 1 | | | | | |
| $pt$ | | | | | | 1 | | | |
| $ss$ | | | | | 2 | | | | |
| $mv$ | | | | | | | | 1 | |
| $nv$ | | | | | | | | | 1 |
| $rt$ | | | | | | | | 1 | |
| | 6 derivs. | | | 6 derivs. | | | | 3 derivs. | |

Table No. 99 (concluded).

| Deg. | 20 | | | | 21 | | 22 | | | 24 | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Ord. | | 0 | 2 | 4 | | 1 | | 0 | 2 | | 0 |
| | | $g^5$<br>$g^3q$<br>$g^2u$<br>$gq^2$<br>$qu$ | $bw$<br>$g^3r$<br>$g^2jo$<br>. . $gjt$<br>$gqr$<br>. . $joq$<br>. . $ov$<br>$ru$ | $b^2g^4$<br>$b^2g^2q$<br>$b^2gu$<br>$b^2q^2$<br>$bg^3m$<br>$bg^2j^2$<br>$bgmq$<br>. . $bgo^2$<br>$bj^2q$<br>. . $bjv$<br>$bmu$<br>. . $bot$<br>$dg^3q$<br>$dgjq$<br>. . $dgv$<br>$dju$<br>$g^4h$<br>$g^2hq$<br>. . $g^2ko$<br>. . $g^2m^2$<br>$ghu$<br>. . $gj^2m$<br>. . $gkt$<br>. . $gos$<br>. . $gr^2$<br>$hq^2$<br>. . $j^4$<br>. . $jor$<br>. . $koq$<br>. . $m^2q$<br>. . $mo^2$<br>. . $st$ | | $g^4j$<br>$g^2jq$<br>$g^2v$<br>$gju$<br>$jq^2$<br>$qv$ | | $gw$ | $bg^5$<br>$bg^3q$<br>$bg^2u$<br>$bgq^2$<br>$bqu$<br>$g^4m$<br>$g^3j^2$<br>$g^2mq$<br>. . $g^2o^2$<br>$gj^2q$<br>. . $gjv$<br>$gmu$<br>. . $got$<br>$j^2u$<br>$mq^2$<br>. . $o^2q$<br>. . $t^2$ | | $g^6$<br>$g^4q$<br>$g^3u$<br>$g^2q^2$<br>$gqu$<br>$q^3$<br>$u^2$ |
| | $ov$ | 1 | | | $ro$ | 1 | $sv$ | | 1 | $tv$ | 1 |
| | $pv$ | | | 1 | | | | | | | |
| | $sv$ | | 1 | | | | | | | | |
| | 3 derivs. | | | | 1 deriv. | | 1 deriv. | | | 1 deriv. | |

384. The Canonical form (using the divided expressions, Table No. 98) is peculiarly convenient for the calculation of the derivatives. Some attention is required in regard to the numerical determination: it will be observed that A is given in the standard form $(A_0, A_1, A_2, A_3, A_4, A_5 \between x, y)^5$, while the other covariants are given in the denumerate forms $B=(B_0, B_1, B_2 \between x, y)^2$ &c.: these must be converted into the other form $B=(B_0, \frac{1}{2}B_1, B_2 \between x, y)^2$, $C=(C_0, \frac{1}{6}C_1, \frac{1}{15}C_2, \frac{1}{20}C_3, \frac{1}{15}C_4, \frac{1}{6}C_5, C_6 \between x, y)^6$, &c., the numerical coefficients being of course the reciprocals of the binomial coefficients. We thus have, for instance, the leading coefficients,

$$l.c. \text{ of } AC2 = A_0.\tfrac{1}{15}C_2 - 2.\ A_1.\tfrac{1}{6}C_1 + A_2.C_0, \quad \text{but}$$
$$\text{,, ,, } BC2 = B_0.\tfrac{1}{15}C_2 - 2.\tfrac{1}{2}B_1.\tfrac{1}{6}C_1 + B_2.C_0.$$

Moreover, as regards the covariants AA2, AA4, &c., we take what are properly the half-values,

*l.c.* of $\mathrm{AA2}=\mathrm{A}_0\mathrm{A}_2-\mathrm{A}_1^2$ (instead of $\mathrm{A}_0\mathrm{A}_2-2\mathrm{A}_1\mathrm{A}_1+\mathrm{A}_2\mathrm{A}_0$),
„ „ $\mathrm{AA4}=\mathrm{A}_0\mathrm{A}_4-4\mathrm{A}_1\mathrm{A}^3+3\mathrm{A}_2^2$ (instead of $\mathrm{A}_0\mathrm{A}_4-4\mathrm{A}_1\mathrm{A}_3+6\mathrm{A}_2\mathrm{A}_2-4\mathrm{A}_3\mathrm{A}_1-\mathrm{A}_4\mathrm{A}_0$),
&c.,

and similarly

*l.c.* of $\mathrm{BB2}=\mathrm{B}_0\mathrm{B}_2-(\tfrac{1}{2}\mathrm{B}_1)^2$
„ „ $\mathrm{CC2}=\mathrm{C}_0.\tfrac{1}{15}\mathrm{C}_2-(\tfrac{1}{6}\mathrm{C}_1)^2$
&c.

Any one of these leading coefficients, for instance *l.c.* of AC2, is equal to the corresponding covariant derivative, multiplied, it may be, by a power of $a$: the index of this power being at once found by comparing the deg-orders, these in fact differing by a multiple of 1.5 the deg-order of $a$. Thus

$aa2$, $\mathrm{A}_0\mathrm{A}_2-\mathrm{A}_1^2$ deg-orders are 2.6, 2.6 or $aa2=\mathrm{A}_0\mathrm{A}_2-\mathrm{A}_1^2$,
$aa4$, $\mathrm{A}_0\mathrm{A}_4-4\mathrm{A}_1\mathrm{A}_3+3\mathrm{A}_2^2$, deg-orders are 2·2, 4·12 or $aa4=\dfrac{1}{a^2}(\mathrm{A}_0\mathrm{A}_4-4\mathrm{A}_1\mathrm{A}_3+3\mathrm{A}_2^2)$;

we have in fact

$$\mathrm{A}_0\mathrm{A}_2-\mathrm{A}_1^2=1.c-0^2=c \quad : \quad \text{and } aa2=c,$$
$$\mathrm{A}_0\mathrm{A}_4-4\mathrm{A}_1\mathrm{A}_3+3\mathrm{A}_2^2=1\ .\ (a^2b-3c^2)-4\ .\ 0\ .f+3\ .\ c^2,=a^2b \quad : \quad \text{and } aa4=b.$$

As another instance, and for the purpose of showing how the calculation is actually effected, consider the derivative $ch2$, which is to be calculated from the leading coefficient of CH2, $=\mathrm{C}_0.\tfrac{1}{6}\mathrm{H}_2-2.\tfrac{1}{6}\mathrm{C}_1.\tfrac{1}{4}\mathrm{H}_1+\tfrac{1}{15}\mathrm{C}_2.\mathrm{H}_0$: this is

$$\begin{aligned} = \ & c(\tfrac{1}{6}a^2g-2abd-ch) \\ & -2\ .\ \tfrac{1}{2}f(\tfrac{1}{2}be-l) \\ & +(\tfrac{1}{5}a^2b-c^2)h \end{aligned}$$

=column next written down; but this column contains congregate terms which have to be replaced by their segregate values (see Table No. 96, deg-order 8·16); and we thus obtain

| | $a^3j$ | $a^2b^3$ | $a^2bh$ | $a^2cg$ | $abcd$ | $b^2c^2$ | $c^2h$ |
|---|---|---|---|---|---|---|---|
| $\tfrac{1}{5}a^2bh$ | | | $+\tfrac{1}{5}$ | | | | |
| $+\tfrac{1}{6}a^2cg$ | | | | $+\tfrac{1}{6}$ | | | |
| $-2abcd$ | | | | | $-2$ | | |
| $-\tfrac{1}{2}bef$ | | | $-\tfrac{1}{2}$ | | $+3$ | $+2$ | |
| $-2c^2h$ | | | | | | | $-2$ |
| $+fl$ | $\tfrac{1}{3}$ | $-\tfrac{1}{3}$ | $+\tfrac{2}{3}$ | $-\tfrac{1}{3}$ | $-1$ | $-2$ | $+2$ |
| $=$ | $\tfrac{1}{3}$ | $-\tfrac{1}{3}$ | $+\tfrac{11}{30}$ | $-\tfrac{1}{6}$ | 0 | 0 | 0 |

viz., the terms other than those divisible by $a^2$ all disappear: we may either abbreviate the calculation by omitting them *ab initio*, or retain them for the sake of the verification afforded by their disappearance. The factor $a^2$ divides out, and the final result is

$$ch2 = \tfrac{1}{3}aj - \tfrac{1}{3}b^3 + \tfrac{11}{30}bh - \tfrac{1}{6}cg,$$

which is the proper segregate expression of the derivative $ch2$: of course, we have deg-order CH2 = 8.16, deg-order $ch2$ = 6.6, and the difference is 2.10, the double of 1.5, so that the factor $a^2$ is as it ought to be.

### Table 100 (The Derivatives up to the Sixth Order).

#### Degree 2.

| 2.2 | $b$ |
|---|---|
| $aa4$ | $+1$ |

| 2.6 | $c$ |
|---|---|
| $aa2$ | $+1$ |

#### Degree 3.

| 3.1 | |
|---|---|
| $ac5$ | 0 |

| 3.3 | $d$ |
|---|---|
| $ab2$ | $-3$ |
| $ac4$ | $+\frac{12}{5}$ |

| 3.5 | $e$ |
|---|---|
| $ab1$ | $+\frac{1}{2}$ |
| $ac3$ | $+\frac{1}{20}$ |

| 3.7 | $ab$ |
|---|---|
| $ac2$ | $+\frac{1}{5}$ |

| 3.9 | $f$ |
|---|---|
| $ac1$ | $+\frac{1}{2}$ |

#### Degree 4.

| 4.0 | $g$ |
|---|---|
| $ae5$ | $-2$ |
| $bb2$ | $-\frac{1}{4}$ |
| $cc6$ | $-\frac{1}{40}$ |

| 4.2 | |
|---|---|
| $ad3$ | 0 |
| $ae4$ | 0 |

| 4.4 | $b^2$ | $h$ |
|---|---|---|
| $ad2$ | $-\frac{1}{3}$ | $+\frac{1}{3}$ |
| $ae3$ | $-\frac{4}{5}$ | $-\frac{6}{5}$ |
| $af5$ | $+\frac{62}{63}$ | $-\frac{83}{63}$ |
| $bc2$ | $+\frac{1}{5}$ | $-\frac{1}{2}$ |
| $cc4$ | $+\frac{3}{25}$ | $-\frac{1}{10}$ |

| 4.6 | $i$ |
|---|---|
| $ad1$ | $+\frac{1}{3}$ |
| $ae2$ | $+\frac{6}{5}$ |
| $af4$ | $+\frac{13}{63}$ |
| $bc1$ | $+\frac{1}{2}$ |

| 4.8 | $ad$ | $bc$ |
|---|---|---|
| $ae1$ | $-\frac{6}{5}$ | $-2$ |
| $af3$ | $+\frac{59}{42}$ | $-\frac{5}{6}$ |
| $cc2$ | $+\frac{1}{4}$ | $-\frac{1}{20}$ |

| 4.10 | $ae$ |
|---|---|
| $af2$ | $+1$ |

| 4.12 | $a^2b$ | $c^2$ |
|---|---|---|
| $af1$ | $+\frac{2}{9}$ | $-2$ |

## Table No. 100 (concluded).

### Degree 5.

| 5.1 | *j* |
|---|---|
| *ah* 4 | $+2$ |
| *ai* 5 | $-\frac{4}{3}$ |
| *bd* 2 | $-\frac{1}{3}$ |
| *ce* 5 | $-\frac{8}{5}$ |

| 5.3 | *k* |
|---|---|
| *ah* 3 | $+\frac{1}{2}$ |
| *ai* 4 | $+\frac{1}{3}$ |
| *bd* 1 | $-\frac{1}{6}$ |
| *be* 2 | $-\frac{3}{5}$ |
| *cd* 3 | $+\frac{3}{20}$ |
| *ce* 4 | $+\frac{2}{25}$ |
| *cf* 6 | $-\frac{19}{42}$ |

| 5.5 | *ag* | *bd* |
|---|---|---|
| *ah* 2 | $+\frac{1}{6}$ | $-2$ |
| *ai* 3 | $+0$ | $-2$ |
| *be* 1 | $-\frac{1}{2}$ | $+\frac{24}{5}$ |
| *cd* 2 | $0$ | $-\frac{2}{15}$ |
| *ce* 3 | $-\frac{1}{20}$ | $-\frac{48}{5}$ |
| *df* 5 | $-\frac{1}{72}$ | $+\frac{8}{63}$ |

| 5.7 | *be* | *l* |
|---|---|---|
| *ah* 1 | $-2$ | $-4$ |
| *ai* 2 | $0$ | $+\frac{1}{3}$ |
| *bf* 2 | $-\frac{7}{36}$ | $+1$ |
| *cd* 1 | $0$ | $+\frac{1}{6}$ |
| *ce* 2 | $\frac{1}{5}$ | $-\frac{2}{5}$ |
| *df* 4 | $-\frac{1}{90}$ | $+\frac{43}{315}$ |

| 5.9 | $ab^2$ | *ah* | *cd* |
|---|---|---|---|
| *ai* 1 | $-\frac{1}{3}$ | $+\frac{1}{3}$ | $+3$ |
| *bf* 1 | $+\frac{2}{9}$ | $-\frac{1}{2}$ | $+3$ |
| *ce* 1 | $0$ | $-\frac{1}{2}$ | $+\frac{9}{5}$ |
| *df* 3 | $+\frac{2}{15}$ | $-\frac{11}{120}$ | $-\frac{439}{420}$ |

| 5.11 | *ai* | *ce* |
|---|---|---|
| *df* 2 | $+\frac{1}{45}$ | $+\frac{1}{180}$ |

| 5.13 | $a^2d$ | *abc* |
|---|---|---|
| *df* 1 | $+\frac{1}{2}$ | $-\frac{5}{8}$ |

### Degree 6.

| 6.0 | |
|---|---|
| *ci* 6 | $0$ |

| 6.2 | *bg* | *m* |
|---|---|---|
| *ak* 3 | $0$ | $-4$ |
| *al* 5 | $0$ | $+\frac{20}{7}$ |
| *bh* 2 | $-\frac{1}{3}$ | $-2$ |
| *ch* 4 | $\frac{1}{10}$ | $-\frac{2}{5}$ |
| *ci* 5 | $0$ | $+\frac{2}{3}$ |
| *dd* 2 | $0$ | $+\frac{1}{9}$ |
| *de* 3 | $0$ | $-\frac{4}{5}$ |
| *ee* 4 | $-1$ | $-\frac{48}{25}$ |
| *ff* 8 | $-\frac{5}{324}$ | $-\frac{68}{567}$ |

| 6.4 | *n* |
|---|---|
| *aj* 1 | $-1$ |
| *ak* 2 | $+\frac{1}{3}$ |
| *al* 4 | $-\frac{1}{35}$ |
| *bh* 1 | $+\frac{1}{2}$ |
| *bi* 2 | $+\frac{1}{3}$ |
| *ch* 3 | $+\frac{3}{10}$ |
| *ci* 4 | $+\frac{1}{15}$ |
| *de* 2 | $-\frac{1}{3}$ |
| *ef* 5 | $-\frac{64}{63}$ |

| 6.6 | *aj* | $b^3$ | *bh* | *cg* |
|---|---|---|---|---|
| *a* 1 | $-\frac{2}{3}$ | $+2$ | $-3$ | $+1$ |
| *al* 3 | $-\frac{16}{35}$ | $+\frac{2}{7}$ | $-\frac{5}{7}$ | $+\frac{3}{7}$ |
| *bi* 1 | $-\frac{1}{3}$ | $+\frac{5}{6}$ | $-\frac{1}{2}$ | $0$ |
| *ch* 2 | $+\frac{1}{3}$ | $-\frac{1}{3}$ | $+\frac{11}{36}$ | $-\frac{1}{6}$ |
| *ci* 3 | $-\frac{1}{6}$ | $-\frac{1}{30}$ | $+\frac{11}{3}$ | $+\frac{7}{60}$ |
| *de* 1 | $-\frac{2}{15}$ | $+\frac{2}{15}$ | $+\frac{1}{15}$ | $-\frac{1}{5}$ |
| *df* 3 | $+\frac{59}{378}$ | $-\frac{143}{378}$ | $+\frac{425}{756}$ | $-\frac{139}{756}$ |
| *ee* 2 | $-\frac{4}{25}$ | $+\frac{4}{25}$ | $-\frac{38}{25}$ | $+\frac{9}{25}$ |
| *ef* 4 | $+\frac{236}{315}$ | $-\frac{4}{105}$ | $+\frac{10}{63}$ | $-\frac{71}{105}$ |
| *ff* 6 | $-\frac{1529}{7938}$ | $+\frac{2873}{7938}$ | $+\frac{3533}{15876}$ | $-\frac{5591}{31752}$ |

| 6.8 | *ak* | *bi* |
|---|---|---|
| *al* 2 | $-\frac{4}{21}$ | $+\frac{1}{21}$ |
| *ch* 1 | $+\frac{1}{6}$ | $+\frac{1}{3}$ |
| *ci* 2 | $+\frac{1}{9}$ | $+\frac{4}{45}$ |
| *df* 2 | $-\frac{11}{108}$ | $-\frac{1}{54}$ |
| *ef* 3 | $+\frac{32}{315}$ | $+\frac{64}{315}$ |

| 6.10 | $a^2g$ | *abd* | $b^2c$ | *ch* |
|---|---|---|---|---|
| *al* 1 | $0$ | $-\frac{3}{7}$ | $-1$ | $+1$ |
| *ci* 1 | $0$ | $-\frac{1}{2}$ | $+\frac{1}{6}$ | $-\frac{1}{6}$ |
| *df* 1 | $0$ | $-\frac{1}{9}$ | $+\frac{1}{3}$ | $-\frac{1}{3}$ |
| *ef* 2 | $+\frac{1}{36}$ | $+\frac{7}{5}$ | $-\frac{19}{45}$ | $-\frac{4}{5}$ |
| *ff* 4 | $+\frac{1}{432}$ | $-\frac{53}{756}$ | $+\frac{89}{756}$ | $-\frac{8}{63}$ |

| 6.12 | *abe* | *al* | *ci* |
|---|---|---|---|
| *ef* 1 | $\frac{2}{9}$ | $-\frac{2}{5}$ | $-\frac{6}{5}$ |

| 6.14 | $a^2b^2$ | $a^2h$ | *acd* | $bc^2$ |
|---|---|---|---|---|
| *ff* 2 | $-\frac{4}{81}$ | $+\frac{1}{36}$ | $+\frac{5}{6}$ | $-\frac{2}{9}$ |

which is complete to the sixth degree. I had calculated the derivatives up to the tenth degree, but the results were not in the segregate form.

### *On the form of the Numerical Generating Functions: the N.G.F. of a Sextic.*
### Article Nos. 385, 386.

385. It is to be remarked that the R.G.F. is derived not from the fraction in its least terms, which is algebraically the most simple form of the N.G.F., but from a form which contains common factors in the numerator and denominator: thus for the quadric, the cubic, and the quartic, writing down the two forms (identical in the case of the quadric) these are—

Quadric

$$\text{N.G.F.}=\frac{1}{1-ax^2 .\ 1-a^2}$$

Cubic

$$\text{N.G.F.}=\frac{1-ax+a^2x^2}{1-a^4 .\ 1-ax^3 .\ 1-ax} \qquad =\frac{1-a^6x^6}{1-a^4 .\ 1-ax^3 .\ 1-a^2x^2 .\ 1-a^3x^3 .}$$

Quartic

$$\text{N.G.F.}=\frac{1-ax^2+a^2x^4}{1-a^2 .\ 1-a^3 .\ 1-ax^4 .\ 1-ax^2} \qquad =\frac{1-a^6x^{12}}{1-a^2 .\ 1-a^3 .\ 1-ax^4 .\ 1-a^2x^2 .\ 1-a^3x^6 .}$$

For the quintic the two forms are, N.G.F. =

| | | | | | | |
|---|---|---|---|---|---|---|
| $($ 1 | | | $-a^6$ | | | $+a^{12})x^0$ |
| $+(-1$ | | $+a^4$ | $+2a^6$ | | | $-a^{12})ax^1$ |
| $+($ | $+a^2$ | | | $-a^8$ | | $+a^{10})x^2$ |
| $+(-1$ | | $+a^4$ | $+a^6$ | $+a^8$ | $-a^{10}$ | $-a^{12})ax^3$ |
| $+(+1$ | $+a^2$ | $-a^4$ | $-a^6$ | $-a^8$ | | $+a^{12})a^2x^4$ |
| $+($ | $-a^2$ | $+a^4$ | | | $-a^{10}$ | $)a^3x^5$ |
| $+(+1$ | | | $-2a^6$ | $-a^8$ | | $+a^{12})a^2x^6$ |
| $+(-1$ | | | $+a^6$ | | | $-a^{12})a^3x^7$ |

divided by

$$1-a^4 .\ 1-a^6 .\ 1-a^8 .\ 1-ax^5 .\ 1-ax^3 .\ 1-ax$$

and

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $(\ 1$ | | | | | | | | | $+a^{18})x^0$ |
| $($ | | $a^4$ | $+a^6$ | | $+a^{10}$ | $+a^{12}$ | | | $)ax$ |
| $($ | | $a^4$ | $+a^6$ | $+a^8$ | $+a^{10}$ | | $+a^{14}$ | | $-a^{18})a^2x^2$ |
| $(\ 1$ | $+a^2$ | $+a^4$ | | $+a^8$ | | | | | $)a^3x^3$ |
| $(\ 1$ | $+a^2$ | $+a^4$ | $+a^6$ | | $+a^{10}$ | | $-a^{14}$ | | $)a^4x^4$ |
| $(\ 1$ | | $+a^4$ | $+a^6$ | | | | | $-a^{16}$ | $)a^3x^5$ |
| $($ | $a^2$ | | | | | $-a^{12}$ | $-a^{14}$ | | $)a^2x^6$ |
| $($ | | $a^4$ | | $-a^8$ | | $-a^{12}$ | $-a^{14}$ | $-a^{16}$ | $-a^{18})ax^7$ |
| $($ | | | | | $-a^{10}$ | $-a^{12}$ | $-a^{14}$ | $-a^{16}$ | $-a^{18})a^2x^8$ |
| $($ | | $-a^4$ | | $-a^8$ | $-a^{10}$ | $-a^{12}$ | $-a^{14}$ | | $)a^3x^9$ |
| $($ | | | $-a^6$ | $-a^8$ | | $-a^{12}$ | $-a^{14}$ | | $)a^4x^{10}$ |
| $(-1$ | | | | | | | | | $-a^{18})a^5x^{11}$ |

divided by

$$1-a^4 .\ 1-a^8 .\ 1-a^{12} .\ 1-ax^5 .\ 1-a^2x^2 .\ 1-a^2x^6$$

this last being in fact equivalent to that used for the determination of the R.G.F.

386. For the sextic the forms are, N.G.F. =

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $(\ 1$ | $+a$ | | $-a^3$ | $-a^4$ | $-a^5$ | | $+a^7$ | $+a^8)x^0$ |
| $(-1$ | $-a$ | $+a^2$ | $+2a^3$ | $+2a^4$ | $+a^5$ | | $-a^7$ | $-a^8)ax^2$ |
| $(-1$ | | $+a^2$ | $+a^3$ | $+a^4$ | $+a^5$ | | $-a^7$ | $-a^8)ax^4$ |
| $(\ 1$ | $+a$ | | $-a^3$ | $-a^4$ | $-a^5$ | $-a^6$ | | $+a^8)a^2x^6$ |
| $(\ 1$ | $+a$ | | $-a^3$ | $-2a^4$ | $-2a^5$ | $-a^6$ | $+a^7$ | $+a^8)a^2x^8$ |
| $(-1$ | $-a$ | | $+a^3$ | $+a^4$ | $+a^5$ | | $-a^7$ | $-a^8)a^3x^{10}$ |

divided by

$$1+a .\ 1-a^2 .\ 1-a^3 .\ 1-a^4 .\ 1-a^5 .\ 1-ax^6 .\ 1-ax^4 .\ 1-ax^2$$

and

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | | | | | | | | | | | | | | $+a^{15}$ | $)x^0$ |
| $+($ 1 | | $+a^2$ | | $+a^4$ | $+a^5$ | | $+a^7$ | | $+a^9$ | | | | | | | $)a^3x^2$ |
| $+($ | | $+a^2$ | $+a^3$ | $+a^4$ | $+a^5$ | $+a^6$ | $+a^7$ | $+a^8$ | $+a^9$ | | $+a^{11}$ | | | | | $)a^2x^4$ |
| $+($ 1 | $+a$ | | $+2a^3$ | | $+a^5$ | $+a^6$ | | $+a^8$ | | | | | $-a^{13}$ | | | $)a^3x^6$ |
| $+($ $+a^{\cdot 5}$ | | $+a^{2\cdot 5}$ | | $+a^{4\cdot 5}$ | | | | | | $-a^{10\cdot 5}$ | | $-a^{12\cdot 5}$ | | $-a^{14\cdot 5}$ | | $)a^{2\cdot 5}x^8$ |
| $+($ | | $+a^2$ | | | | | $-a^7$ | | $-a^9$ | $-a^{10}$ | | $-2a^{12}$ | | $-a^{14}$ | $-a^{15}$ | $)a^2x^{10}$ |
| $+($ | | | | $-a^4$ | | $-a^6$ | $-a^7$ | $-a^8$ | $-a^9$ | $-a^{10}$ | $-a^{11}$ | $-a^{12}$ | $-a^{13}$ | | | $)a^3x^{12}$ |
| $+($ | | | | | | $-a^6$ | | $-a^8$ | | $-a^{10}$ | $-a^{11}$ | | $-a^{13}$ | | $-a^{15}$ | $)a^2x^{14}$ |
| $+(-1$ | | | | | | | | | | | | | | | $-a^{15}$ | $)a^5x^{16}$ |

divided by

$$1-a^2 . 1-a^4 . 1-a^6 . 1-a^{10} . 1-ax^6 . 1-a^2x^4 . 1-a^2x^8$$

where observe that in the middle term, although for symmetry $a^{\cdot 5}$ $(=\sqrt{a})$ has been introduced into the expression, the coefficient is really rational, viz., the term is $(a^3+a^5+a^7-a^{13}-a^{15}-a^{17})x^8$. The second form or one equivalent to it is due to SYLVESTER: I do not know whether he divided out the common factors so as to obtain the first form. I assume that it would be possible from this second form to obtain a R.G.F., and thence to establish for the 26 covariants of the sextic a theory such as has been given for the 23 covariants of the quintic, but I have not entered upon this question.

## Table 93 *bis* (The covariant S, adopted form $=-(\mathrm{D}, \mathrm{M})$).

In this Table, $a, b, c, d, e, f$ denote, as in the tables of former memoirs, the coefficients of the quintic form $(a, b, c, d, e, f \between x, y)^5$.

$\mathrm{S}=($

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $a^3b^0c^3f^3$ | $-\ \ 2$ | $a^3b^0c^2df^3$ | $-\ \ 3$ | $a^3b^0cd^2f^3$ | $+\ \ 3$ | $a^3b^0d^3f^3$ | $+\ \ 2$ |
| $c^2def^2$ | $+\ 15$ | $c^2e^2f^2$ | $+\ \ 3$ | $cde^2f^2$ | $-\ \ 6$ | $d^2e^2f^2$ | $-\ \ 6$ |
| $c^2e^3f$ | $-\ \ 9$ | $cd^2ef^2$ | $+\ 24$ | $ce^4f$ | $+\ \ 3$ | $de^4f$ | $+\ \ 6$ |
| $cd^3f^2$ | $-\ \ 9$ | $cde^3f$ | $-\ 42$ | $d^3ef^2$ | $-\ \ 3$ | $e^6$ | $-\ \ 2$ |
| $cd^2e^2f$ | $-\ \ 6$ | $ce^5$ | $+\ 18$ | $d^2e^3f$ | $+\ \ 6$ | $a^2b\ cd^2f^3$ | $-\ 15$ |
| $cde^4$ | $+\ \ 9$ | $d^4f^2$ | $-\ 18$ | $de^5$ | $-\ \ 3$ | $cde^2f^2$ | $+\ 30$ |
| $d^4ef$ | $+\ \ 9$ | $d^3e^2f$ | $+\ 33$ | $a^2b^2d^2f^3$ | $-\ \ 3$ | $ce^4f$ | $-\ 15$ |
| $d^3e^3$ | $-\ \ 7$ | $d^2e^4$ | $-\ 15$ | $de^2f^2$ | $+\ \ 6$ | $d^3ef^2$ | $+\ 15$ |
| $a^2b^2c^2f^3$ | $+\ \ 6$ | $a^2b^2cdf^3$ | $+\ \ 6$ | $e^4f$ | $-\ \ 3$ | $d^2e^3f$ | $-\ 30$ |
| $cdef^2$ | $-\ 30$ | $ce^2f^2$ | $-\ \ 6$ | ,, $b\ c^2df^3$ | $-\ 24$ | $de^5$ | $+\ 15$ |
| $ce^3f$ | $+\ 18$ | $d^2ef^2$ | $-\ 24$ | $c^2e^2f^2$ | $+\ 24$ | ,, $b^0c^3df^3$ | $+\ \ 9$ |
| $d^3f^2$ | $+\ \ 9$ | $de^3f$ | $+\ 42$ | $cd^2ef^2$ | $+\ 78$ | $c^3e^2f^2$ | $-\ \ 9$ |
| $d^2e^2f$ | $+\ \ 6$ | $e^5$ | $-\ 18$ | $cde^3f$ | $-108$ | $c^2d^2ef^2$ | $-\ 21$ |
| $de^4$ | $-\ \ 9$ | ,, $b\ c^3f^3$ | $+\ \ 3$ | $ce^5$ | $+\ 30$ | $c^2de^3f$ | $+\ 15$ |
| ,, $b\ c^3ef^2$ | $-\ 15$ | $c^2def^2$ | $-\ 78$ | $d^4f^2$ | $-\ 24$ | $c^2e^5$ | $+\ \ 6$ |
| $c^2d^2f^2$ | $+\ 21$ | $c^2e^3f$ | $+\ 69$ | $d^3e^2f$ | $+\ 24$ | $cd^4f^2$ | $+\ \ 3$ |
| $c^2de^2f$ | $-\ \ 6$ | $cd^3f^2$ | $+\ 93$ | ,, $b^0c^4f^3$ | $+\ 18$ | $cd^3e^2f$ | $+\ 21$ |
| $c^2e^4$ | $+\ 18$ | $cd^2e^2f$ | $-\ 51$ | $c^3def^2$ | $-\ 93$ | $cd^2e^4$ | $-\ 24$ |
| $cd^3ef$ | $+\ 30$ | $cde^4$ | $-\ 33$ | $c^3e^3f$ | $+\ 21$ | $d^5ef$ | $-\ \ 9$ |
| $cd^2e^3$ | $-\ 51$ | $d^4ef$ | $-\ 57$ | $c^2d^3f^2$ | $+\ 36$ | $d^4e^3$ | $+\ \ 9$ |
| $d^5f$ | $-\ 36$ | $d^3e^3$ | $+\ 54$ | $c^2d^2e^2f$ | $+123$ | $a\ b^3d^2f^3$ | $+\ \ 9$ |
| $d^4e^2$ | $+\ 39$ | ,, $b^0c^4ef^2$ | $+\ 24$ | $c^2de^4$ | $-\ 51$ | $de^2f^2$ | $-\ 18$ |
| ,, $b^0c^4df^2$ | $-\ \ 3$ | $c^3d^2f^2$ | $-\ 36$ | $cd^4ef$ | $-111$ | $e^4f$ | $+\ \ 9$ |
| $c^4e^2f$ | $+\ 45$ | $c^3de^2f$ | $-\ \ 9$ | $cd^3e^3$ | $+\ 39$ | ,, $b^2c^2df^3$ | $+\ \ 6$ |
| $c^3d^2ef$ | $-\ 84$ | $c^3e^4$ | $-\ 54$ | $d^6f$ | $+\ 27$ | $c^2e^2f^2$ | $-\ \ 6$ |
| $c^3de^3$ | $-\ 63$ | $c^2d^3ef$ | $+\ 24$ | $d^5e^2$ | $-\ \ 9$ | $cd^2ef^2$ | $+\ \ 6$ |
| $c^2d^4f$ | $+\ 45$ | $c^2d^2e^3$ | $+129$ | $a\ b^3cdf^3$ | $+\ 42$ | $cde^3f$ | $-\ 24$ |
| $c^2d^3e^2$ | $+150$ | $cd^5f$ | $+\ \ 9$ | $ce^2f^2$ | $-\ 42$ | $ce^5$ | $+\ 18$ |
| $cd^5e$ | $-117$ | $cd^4e^2$ | $-114$ | $d^2ef^2$ | $-\ 69$ | $d^4f^2$ | $-\ 45$ |
| $d^7$ | $+\ 27$ | $d^6e$ | $+\ 27$ | $de^3f$ | $+\ 96$ | $d^3e^2f$ | $+\ 96$ |
| $a^1b^4cf^3$ | $-\ \ 6$ | $a^1b^4df^3$ | $-\ \ 3$ | $e^5$ | $-\ 27$ | $d^2e^4$ | $-\ 51$ |
| $def^2$ | $+\ 15$ | $e^2f^2$ | $+\ \ 3$ | ,, $b^2c^3f^3$ | $-\ 33$ | ,, $b\ c^4f^3$ | $-\ \ 9$ |
| $e^3f$ | $-\ \ 9$ | ,, $b^3c^2f^3$ | $-\ \ 6$ | $c^2def^2$ | $+\ 51$ | $c^3def^2$ | $-\ 30$ |
| ,, $b^3c^2ef^2$ | $+\ 30$ | $cdef^2$ | $+108$ | $c^2e^3f$ | $+\ 48$ | $c^3e^3f$ | $+\ 66$ |
| $cd^2f^2$ | $-\ 15$ | $ce^3f$ | $-\ 96$ | $cd^3f^2$ | $+\ \ 9$ | $c^2d^3f^2$ | $+\ 84$ |
| $cde^2f$ | $+\ 24$ | $d^3f^2$ | $-\ 21$ | $cd^2e^2f$ | $-147$ | $c^2d^2e^2f$ | $-\ 36$ |
| $ce^4$ | $-\ 45$ | $d^2e^2f$ | $-\ 48$ | $cde^4$ | $+\ 39$ | $c^2de^4$ | $-102$ |
| $d^3ef$ | $-\ 66$ | $de^4$ | $+\ 63$ | $d^4ef$ | $+\ 78$ | $cd^4ef$ | $-174$ |
| $d^2e^3$ | $+\ 72$ | ,, $b^2c^3ef^2$ | $-\ 24$ | $d^3e^3$ | $-\ 45$ | $cd^3e^3$ | $+210$ |
| ,, $b^2c^3df^2$ | $-\ 21$ | $c^2d^2f$ | $-123$ | ,, $b\ c^4ef^2$ | $+\ 57$ | $d^6f$ | $+\ 63$ |
| $c^3e^2f$ | $-\ 96$ | $c^2de^2f$ | $+147$ | $c^3d^2f^2$ | $-\ 24$ | $d^5e^2$ | $-\ 72$ |
| $c^2d^2ef$ | $+\ 36$ | $c^2e^4$ | $+\ 66$ | $c^3de^2f$ | $-\ 78$ | ,, $b^0c^5ef^2$ | $+\ 36$ |
| $c^2de^3$ | $+213$ | $cd^3ef$ | $+\ 78$ | $c^3e^4$ | $-\ 60$ | $c^4d^2f^2$ | $-\ 45$ |
| $cd^4f$ | $+120$ | $cd^2e^3$ | $-186$ | $c^2d^3ef$ | $+\ 36$ | $c^4de^2f$ | $-120$ |
| $cd^3e^2$ | $-303$ | $d^5f$ | $+\ 51$ | $c^2d^2e^3$ | $+108$ | $c^4e^4$ | $-\ \ 6$ |
| $d^5e$ | $+\ 51$ | $d^4e^2$ | $-\ \ 9$ | $cd^5f$ | $-\ 24$ | $c^3d^3ef$ | $+204$ |
| ,, $b\ c^5f^2$ | $+\ \ 9$ | ,, $b\ c^4df^2$ | $+111$ | $cd^4e^2$ | $-\ \ 6$ | $c^3d^2e^3$ | $+120$ |
| $c^4def$ | $+174$ | $c^4e^2f$ | $-\ 78$ | $d^6e$ | $-\ \ 9$ | $c^2d^5f$ | $-\ 66$ |
| $c^4e^3$ | $-\ 36$ | $c^3d^2ef$ | $-\ 36$ | ,, $b^0c^5df^2$ | $-\ \ 9$ | $c^2d^4e^2$ | $-240$ |
| $c^3d^3f$ | $-204$ | $c^3de^3$ | $-\ 54$ | $c^5e^2f$ | $-\ 51$ | $cd^6e$ | $+144$ |
| $c^3d^2e^2$ | $-174$ | $c^2d^4f$ | $-\ 96$ | $c^4d^2ef$ | $+\ 96$ | $d^8$ | $-\ 27$ |
| $c^2d^4e$ | $+330$ | $c^2d^3e^2$ | $+150$ | $c^4de^3$ | $+111$ | $a^0b^4cdf^3$ | $-\ \ 9$ |
| $cd^6$ | $-\ 99$ | $cd^5e$ | $+\ 30$ | $c^3d^4f$ | $-\ 27$ | $ce^2f^2$ | $+\ \ 9$ |

$\between x, y)^3$

(continued on next page.)

(continued from last page.)

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $a^1b^0c^6ef$ | $-\ 63$ | $a^1b\,d^7$ | $-\ 27$ | $a^1b^0c^3d^3e^2$ | $-234$ | $a^0b^4d^2ef^2$ | $-\ 18$ |
| $c^5d^2f$ | $+\ 66$ | ,, $b^0c^6f^2$ | $-\ 27$ | $c^2d^5e$ | $+141$ | $de^3f$ | $+\ 45$ |
| $c^5de^2$ | $+\ 99$ | $c^5def$ | $+\ 24$ | $cd^7$ | $-\ 27$ | $e^5$ | $-\ 27$ |
| $c^4d^3e$ | $-147$ | $c^5e^3$ | $+\ 54$ | $a^0b^5df^3$ | $-\ 18$ | ,, $b^3c^3f^3$ | $+\ 7$ |
| $c^3d^5$ | $+\ 45$ | $c^4d^3f$ | $+\ 27$ | $e^2f^2$ | $+\ 18$ | $c^2def^2$ | $+\ 51$ |
| $a^0b^6f^3$ | $+\ 2$ | $c^4d^2e^2$ | $-\ 93$ | ,, $b^4c^2f^3$ | $+\ 15$ | $c^2e^3f$ | $-\ 72$ |
| $b^5cef^2$ | $-\ 15$ | $c^3d^4e$ | $+\ 6$ | $cdef^2$ | $+\ 33$ | $cd^3f^2$ | $+\ 63$ |
| $d^2f^2$ | $-\ 6$ | $c^2d^6$ | $+\ 9$ | $ce^3f$ | $-\ 63$ | $cd^2e^2f$ | $-213$ |
| $de^2f$ | $-\ 18$ | $a^0b^5cf^3$ | $+\ 3$ | $d^3f^2$ | $+\ 54$ | $cde^4$ | $+171$ |
| $e^4$ | $+\ 27$ | $def^2$ | $-\ 30$ | $d^2e^2f$ | $-\ 66$ | $d^4ef$ | $+\ 36$ |
| ,, $b^4c^2df^2$ | $+\ 24$ | $e^3f$ | $+\ 27$ | $de^4$ | $+\ 27$ | $d^3e^3$ | $-\ 43$ |
| $c^2e^2f$ | $+\ 51$ | ,, $b^4cd^2f^2$ | $+\ 51$ | ,, $b^3c^3ef^2$ | $-\ 54$ | ,, $b^2c^4ef^2$ | $-\ 39$ |
| $cd^2ef$ | $+102$ | $cde^2f$ | $-\ 39$ | $c^2d^2f^2$ | $-129$ | $c^3d^2f^2$ | $-150$ |
| $cde^3$ | $-171$ | $ce^4$ | $-\ 27$ | $c^2de^2f$ | $+186$ | $c^3de^2f$ | $+303$ |
| $d^4f$ | $+\ 6$ | $d^3ef$ | $+\ 60$ | $c^2e^4$ | $+\ 45$ | $c^3e^4$ | $-\ 18$ |
| $d^3e^2$ | $+\ 18$ | $d^2e^3$ | $-\ 45$ | $cd^3ef$ | $+\ 54$ | $c^2d^3ef$ | $+174$ |
| ,, $b^3c^4f^2$ | $-\ 9$ | ,, $b^3c^3df^2$ | $-\ 39$ | $cd^2e^3$ | $-\ 96$ | $c^2d^2e^3$ | $-345$ |
| $c^3def$ | $-210$ | $c^3e^2f$ | $+\ 45$ | $d^5f$ | $-\ 54$ | $cd^5f$ | $-\ 99$ |
| $c^3e^3$ | $+\ 43$ | $c^2d^2ef$ | $-108$ | $d^4e^2$ | $+\ 48$ | $cd^4e^2$ | $+192$ |
| $c^2d^3f$ | $-120$ | $c^2de^3$ | $+\ 96$ | ,, $b^2c^4df^2$ | $+114$ | $d^6e$ | $-\ 18$ |
| $c^2d^2e^2$ | $+345$ | $cd^4f$ | $-111$ | $c^4e^2f$ | $+\ 9$ | ,, $b\,c^5df^2$ | $+117$ |
| $cd^4e$ | $-\ 87$ | $cd^3e^2$ | $+147$ | $c^3d^2ef$ | $-150$ | $c^5e^2f$ | $-\ 51$ |
| $d^6$ | $-\ 2$ | $d^5e$ | $-\ 30$ | $c^3de^3$ | $-147$ | $c^4d^2ef$ | $-330$ |
| ,, $b^2c^5ef$ | $+\ 72$ | ,, $b^2c^5f^2$ | $+\ 9$ | $c^2d^4f$ | $+\ 93$ | $c^4de^3$ | $+\ 87$ |
| $c^4d^2f$ | $+240$ | $c^4def$ | $+\ 6$ | $c^2d^3e^2$ | $+150$ | $c^3d^4f$ | $+147$ |
| $c^4de^2$ | $-192$ | $c^4e^3$ | $-\ 48$ | $cd^5e$ | $-\ 87$ | $c^3d^3e^2$ | $+186$ |
| $c^3d^3e$ | $-186$ | $c^3d^3f$ | $+234$ | $d^7$ | $+\ 18$ | $c^2d^5e$ | $-201$ |
| $c^2d^5$ | $+\ 96$ | $c^3d^2e^2$ | $-150$ | ,, $b\,c^6f^2$ | $-\ 27$ | $cd^7$ | $+\ 45$ |
| ,, $b\,c^6df$ | $-144$ | $c^2d^4e$ | $-108$ | $c^5def$ | $-\ 30$ | ,, $b^0c^7f^2$ | $-\ 27$ |
| $c^6e^2$ | $+\ 18$ | $cd^6$ | $+\ 57$ | $c^5e^3$ | $+\ 30$ | $c^6def$ | $+\ 99$ |
| $c^5d^2e$ | $+201$ | ,, $b\,c^6ef$ | $+\ 9$ | $c^4d^3f$ | $-\ 6$ | $c^6e^3$ | $+\ 2$ |
| $c^4d^4$ | $-\ 87$ | $c^5d^2f$ | $-141$ | $c^4d^2e^2$ | $+108$ | $c^5d^3f$ | $-\ 45$ |
| ,, $b^0c^8f$ | $+\ 27$ | $c^5de^2$ | $+\ 87$ | $c^3d^4e$ | $-\ 96$ | $c^5d^2e^2$ | $-\ 96$ |
| $c^7de$ | $-\ 45$ | $c^4d^3e$ | $+\ 96$ | $c^2d^6$ | $+\ 21$ | $c^4d^4e$ | $+\ 87$ |
| $c^6d^3$ | $+\ 20$ | $c^3d^5$ | $-\ 51$ | ,, $b^0c^7ef$ | $+\ 27$ | $c^3d^6$ | $-\ 20$ |
| | | ,, $b^0c^7df$ | $+\ 27$ | $c^6d^2f$ | $-\ 9$ | | |
| | | $c^7e^2$ | $-\ 18$ | $c^6de^2$ | $-\ 57$ | | |
| | | $c^6d^2e$ | $-\ 21$ | $c^5d^3e$ | $+\ 51$ | | |
| | | $c^5d^4$ | $+\ 12$ | $c^4d^5$ | $-\ 12$ | | |

I remark that I calculated the first two coefficients $S_0$, $S_1$, and deduced the other two $S_2$ from $S_1$, and $S_3$ from $S_0$, by reversing the order of the letters (or which is the same thing, interchanging $a$ and $f$, $b$ and $e$, $c$ and $d$) and reversing also the signs of the numerical coefficients. This process for $S_2$, $S_3$ is to a very great extent a verification of the values of $S_0$, $S_1$. For, as presently mentioned, the terms of $S_0$ form subdivisions such that in each subdivision the sum of the numerical coefficients is $=0$: in passing by the reversal process to the value of $S_3$, the terms are distributed into an entirely new set of subdivisions, and then in each of these subdivisions the sum of the numerical coefficients is found to be $=0$; and the like as regards $S_1$ and $S_3$.

If in the expressions for $S_0$, $S_1$, $S_2$, $S_3$ we first write $d=e=f=1$, thus in effect combining the numerical coefficients for the terms which contain the same powers in $a$, $b$, $c$, we find

$$
\begin{aligned}
\mathrm{S}_0 = {} & a^3(-2c^3+6c^2-6c+2)\\
& +a^2\{b^2(6c^2-12c-6)+b(-15c^3+33c^2-21c+3)\\
& \qquad +b^0(42c^4-147c^3+195c^2-117c+27)\}\\
& +a\ \{b^4.\ 0+b^3(30c^2-36c+6)+b^2(-117c^3+249c^2-183c+51)\\
& \qquad +b(9c^5+138c^4-378c^3+330c^2-99c)+b^0(-63c^6+165c^5-147c^4+45c^3)\}\\
& +a^0.\{b^6.\ 2+b^5(-15c+3)+b^4(75c^2-69c+24)+b^3(-9c^4-167c^3+225c^2-87c-2)\\
& \qquad +b^2(72c^5+48c^4-186c^3+96c^2)+b(-126c^6+201c^5-87c^4)\\
& \qquad +b^0(27c^8-45c^7+20c^6)\}
\end{aligned}
$$

which for $c=1$ becomes

$$=2b^6-12b^5+30b^4-40b^3+30b^2-12b+2, \text{ that is } 2(b-1)^6.$$

and for $b=1$, becomes $=0$.

$$
\begin{aligned}
\mathrm{S}_2 = {} & a^3(0c^2+0c+0)\\
& +a^2\{b^2(0c+0)+b(3c^3-9c^2+9c-3)+b^0(24c^4-99c^3+153c^2-105c+27)\}\\
& +a\ \{b^4.\ 0+b^3(-6c^2+12c-6)+b^2(-24c^3+90c^2-108c+42)\\
& \qquad +b(33c^4-90c^3+54c^2+30c-27)+b^0.(-27c^6+78c^5-66c^4+6c^3+9c^2)\}\\
& +a^0\{b^5(3c-3)+b^4(-15c+15)+b^3(6c^3-12c^2+36c-30)\\
& \qquad +b^2(9c^5-42c^4+84c^3-108c^2+57c)+b(9c^6-54c^5+96c^4-51c^3)\\
& \qquad +b^0(9c^7-9c^6)\}
\end{aligned}
$$

which for $c=1$ becomes $=0$.

$$
\begin{aligned}
\mathrm{S}_3 = {} & a^3(0c+0)\\
& +a^2\{b^2.\ 0+b(0c^2+0c+0)+b^0(18c^4-72c^3+108c^2-72c+18)\}\\
& +a\ \{b^3(0c+0)+b^2(-33c^3+99c^2-99c+33)+b(57c^4-162c^3+144c^2-30c-9)\\
& \qquad +b^0(-60c^5+207c^4-261c^3+141c^2-27c)\}\\
& +a^0\{b^5.\ 0+b^4(15c^2-30c+15)+b^3(-54c^3+102c^2-42c-6)\\
& \qquad +b^2(123c^4-297c^3+243c^2-87c+18)+b(-27c^6+102c^4-96c^3+21c^2)\\
& \qquad +b^0(27c^7-66c^6+51c^5-12c^4)\}
\end{aligned}
$$

which for $c=1$ becomes $=0$.

$$
\begin{aligned}
\mathrm{S}_4 = {} & a^3 .\, 0 \\
& + a^2 \{ b(0c+0) + b^0(0c^3+0c^2+0c+0) \} \\
& + a\ \{ b^3 .\, 0 + b^2(0c^2+0c+0) + b(-9c^4+36c^3-54c^2+36c-9) \\
& \qquad + b^0(36c^5-171c^4+324c^3-306c^2+144c-27) \} \\
& + a^0 \{ b^4(0c+0) + b^3(7c^3-21c^2+21c-7) + b^2(-39c^4+135c^3-171c^2+93c-18) \\
& \qquad + b(66c^5-243c^4+333c^3-201c^2+45c) \\
& \qquad + b^0(-27c^7+101c^6-141c^5+87c^4-20c^3) \}
\end{aligned}
$$

which for $c=1$ becomes $=0$.

It follows that for $c=d=e=f=1$, the value of the covariant S is $=2(b-1)^6x^3$, which might be easily verified.